182

FINAL

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section: F

Math33B
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Problem	Points	Score
1	7	
2	5	
3	10	
4	10	
5	8	
6	9	
7	9	
8	8	
9	9	
10	9	
11	11	
12	5	
Total	100	

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Solve the following equation. (*Hint: Find the integrating factor*)

$$(x^2+y^2)\ dx - 2xy\ dy = 0$$

Let $\mu = \mu(x)$

$$h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{-2xy}(2y+2y) = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\mu(x) = e^{\int h(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$\mu P dx + \mu Q dy$

$$= (1+\frac{y^2}{x^2})dx - (\frac{2y}{x})dy = 0$$

$$F(x,y) = \int 1+\frac{y^2}{x^2}dx = x - \frac{y^2}{x} + \phi(y)$$

$$-\frac{2y}{x} = \frac{\partial(x-\frac{y^2}{x}+\phi(y))}{\partial y} = -\frac{2y}{x} + \phi'(y) \Rightarrow \phi(y) = 0$$

$$\Rightarrow F(x,y) = \boxed{x - \frac{y^2}{x} = c}$$

Exercise 2. (5pt) Solve $y' = y(y+1)(x+2)(x+3)$

$$y' = y(x^2+5x+6) + y^2(x^2+5x+6)$$

let $z = y^{-1}$

$z' = -y^{-2}y'$

multiply both sides by $-y^{-2}$

$$-y^{-2}y' = -y^{-1}(x^2+5x+6) - (x^2+5x+6)$$

$$z' = -z(x^2+5x+6) - (x^2+5x+6)$$

Solve by integration factor:

$$\mu = e^{\int -a(x)\,dx} = e^{\int x^2+5x+6} = e^{\frac{1}{3}x^3+\frac{5}{2}x^2+6x}$$

$$\int (\mu z)' = \int -(e^{\frac{1}{3}x^3+\frac{5}{2}x^2+6x})(x^2+5x+6)$$

$$e^{\frac{1}{3}x^3+\frac{5}{2}x^2+6x} \cdot z = -e^{\frac{1}{3}x^3+\frac{5}{2}x^2+6x} + c$$

$$z = -1 + ce^{-\frac{1}{3}x^3-\frac{5}{2}x^2-6x}$$

$y = \frac{1}{z}$

$$\Rightarrow \boxed{y(x) = \frac{1}{ce^{-\frac{1}{3}x^3-\frac{5}{2}x^2-6x} - 1}}$$

Exercise 3. (10pt) Find a **particular** solution to the following two differential equations

(1) $y'' + 4y = 8t^2 - 4t$ (2pt)

Let $y_p(t) = at^2 + bt + c$

$y_p' = 2at + b \qquad y_p'' = 2a$

$2a + 4(at^2 + bt + c) = 8t^2 - 4t$

$4a = 8 \rightarrow a = 2$

$4b = -4 \rightarrow b = -1$

$2a + 4c = 0 \rightarrow c = -1$

$\Rightarrow \boxed{y_p(t) = 2t^2 - t - 1}$

(2) $y'' + 4y = 4\sin(2t)$ (4pt)

$(4\sin 2t)'' + 4(4\sin 2t)$

$= -16\sin 2t + 16\sin 2t = 0$ so the forcing term solves the homogeneous equation, so we try multiplying with t.

Let $z = ate^{2it}$ and $y_p = \mathrm{Im}(z)$

$z' = ae^{2it} + a \cdot 2i \cdot te^{2it}$

$z'' = 2iae^{2it} + 2iae^{2it} - 4ate^{2it}$

Solve $z'' + 4z = 4e^{2it}$:

$(4iae^{2it} - \cancel{4ate^{2it}}) + 4(\cancel{ate^{2it}}) = 4e^{2it}$

$4iae^{2it} = 4e^{2it}$

$ai = 1$

$a = -i$

$\Rightarrow z = -ite^{2it} = -it(\cos 2t + i\sin 2t) = t\sin 2t - it\cos 2t$

$y_p = \mathrm{Im}(z) = \boxed{-t\cos 2t}$

(3) Give the **general** solution to the following differential equation

$$y'' + 4y = 8\sin(2t) - 8t^2 + 4t. \text{ (4pt)}$$

Solve homogeneous:

$y_h'' + 4y_h = 0$

$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$e^{2it} = e^{0t}(\cos 2t + i\sin 2t) \Rightarrow y_h(t) = c_1\cos 2t + c_2 \sin 2t$

Find particular solution:

$y_1'' + 4y_1 = 8\sin 2t$

From the previous problem, we have $y = -t\cos 2t$ solve $y'' + 4y = 4\sin 2t$

So here we know $y_1 = -2t\cos 2t$ solves $y_1'' + 4y_1 = 8\sin 2t$

$y_2'' + 4y_2 = -8t^2 + 4t$

From the previous problem we have $y = 2t^2 - t - 1$ solves $y'' + 4y = 8t^2 - 4t$

so here we know $y_2 = -2t^2 + t + 1$ solves $y_2'' + 4y_2 = -8t^2 + 4t$

$y(t) = y_p(t) + y_h(t)$

$$\boxed{y(t) = -2t\cos 2t - 2t^2 + t + 1 + c_1\cos 2t + c_2\sin 2t}$$

Exercise 4. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 1 \\ -6 \\ -2 \end{pmatrix}$$

This is an upper triangular matrix, so the eigenvalues are those on the diagonal.

$\lambda = 3, -1$ (with alg mult 3)

$\lambda = 3$:

$$A - 3I = \begin{pmatrix} -4 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 \end{pmatrix} \qquad \text{eigenvector } \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 8 \\ 0 \end{pmatrix}$$

$\lambda = -1$:

$$A + I = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A+I)^2 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 8 & -6 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 64 & -48 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $\vec{v}_3$ be in nullspace of $(A+I)^3$:

$$\vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} \qquad \vec{v}_3 = (A+I)\vec{v}_4 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = (A+I)^2 \vec{v}_4 = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1(t) = e^{3t}\vec{V}_1 = e^{3t}\begin{pmatrix}1\\2\\8\\0\end{pmatrix}$$

$$y_2(t) = e^{-t}\vec{V}_2 = e^{-t}\begin{pmatrix}6\\0\\0\\0\end{pmatrix} \text{ or } e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$y_3(t) = e^{-t}(\vec{V}_3 + t\vec{V}_2) = e^{-t}\left(\begin{pmatrix}0\\3\\0\\0\end{pmatrix} + t\begin{pmatrix}6\\0\\0\\0\end{pmatrix}\right) \text{ or } e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\0\\0\end{pmatrix}\right)$$

$$y_4(t) = e^{-t}(V_4 + tV_3 + \tfrac{t^2}{2}V_2) = e^{-t}\left(\begin{pmatrix}0\\0\\3\\4\end{pmatrix} + t\begin{pmatrix}0\\3\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}6\\0\\0\\0\end{pmatrix}\right)$$

$$\boxed{y(t) = c_1 e^{3t}\begin{pmatrix}1\\2\\8\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_3 e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\0\\0\end{pmatrix}\right) + c_4 e^{-t}\left(\begin{pmatrix}0\\0\\3\\4\end{pmatrix} + t\begin{pmatrix}0\\3\\0\\0\end{pmatrix} + t^2\begin{pmatrix}3\\0\\0\\0\end{pmatrix}\right)}$$

OR

$$\boxed{y(t) = c_1 e^{3t}\begin{pmatrix}1\\2\\8\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}2t\\1\\0\\0\end{pmatrix} + c_4 e^{-t}\begin{pmatrix}3t^2\\3t\\3\\4\end{pmatrix}}$$

$$\vec{y}(0) = \begin{pmatrix}3\\1\\-6\\2\end{pmatrix}$$

$c_1 + c_2 = 3 \rightarrow c_2 = 3 + 15/16 = 63/16$

$2c_1 + c_3 = 1 \rightarrow c_3 = 1 + 15/8 = 23/8$

$8c_1 + 3c_4 = -6 \rightarrow c_1 = \frac{-6 - 3/2}{8} = -15/16$

$4c_4 = 2 \rightarrow c_4 = 1/2$

$$\boxed{y(t) = -\frac{15}{16}e^{3t}\begin{pmatrix}1\\2\\8\\0\end{pmatrix} + \frac{63}{16}e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{23}{8}e^{-t}\begin{pmatrix}2t\\1\\0\\0\end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix}3t^2\\3t\\3\\4\end{pmatrix}}$$

Exercise 5. (8pt) Consider the differential equation

$$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t-1) - t(e^t+1), \qquad (t>0)$$

(1) Show that $y_1 = e^t(2t+1) - (t+1)$ is solutions to the above equation. (4pt)
(Show ALL your calculations in detail for full credit)

$$y_1' = e^t(2t+1) + 2e^t - 1 = e^t(2t+3) - 1$$

$$y_1'' = e^t(2t+1) + 2e^t + 2e^t = e^t(2t+5)$$

$$t^2y_1'' - (t^2+2t)y_1 + (t+2)y_1$$

$$= t^2e^t(2t+5) - (t^2+2t)(e^t(2t+3)-1) + (t+2)(e^t(2t+1)-(t+1))$$

$$= 2t^3e^t + 5t^2e^t - (t^2+2t)(2te^t+3e^t-1) + (t+2)(2te^t+e^t-t-1)$$

$$= \cancel{2t^3e^t} + 5t^2e^t - \cancel{2t^3e^t} - 3t^2e^t + t^2 - 4t^2e^t - 6te^t + 2t + 2t^2e^t + te^t - t^2 - t + 4te^t$$

$$= \underbrace{(5t^2e^t - 3t^2e^t - 4t^2e^t + 2t^2e^t)}_{\to 0} + \cancel{t^2} - 6te^t + 2t + te^t - \cancel{t^2} - t + 4te^t + 2e^t - 2t - 2 \quad + 2e^t - 2t - 2$$

$$= -6te^t \cancel{+2t} + te^t - t + 4te^t + 2e^t \cancel{-2t} - 2$$

$$= -te^t - t + 2e^t - 2$$

$$= 2(e^t-1) - t(e^t+1) \quad \checkmark$$

(2) Given that $y_2 = e^t(t+1)+(t-1)$, and $y_3 = e^t(1-t)+(2t-1)$ are also solutions to the above equation, find the general solution to the equation. Justify your answer. (4pt)

$y_1 = e^t(2t+1) + (-t-1)$

$y_2 = e^t(t+1) + (t-1)$

$y_3 = e^t(-t+1) + (2t-1)$

$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t-1) - t(e^t+1)$

$y_1 - y_2 = te^t - 2t$

$y_2 - y_3 = 2te^t - t$

$y_1 - y_3 = 3te^t - 3t$

Guess: te^t and t are solutions to the homogeneous equation.

For te^t: $(te^t)' = e^t + te^t \quad (te^t)'' = e^t + e^t + te^t = 2e^t + te^t$

$t^2(2e^t + te^t) - (t^2+2t)(e^t+te^t) + (t+2)(te^t)$

$= 2t^2e^t \cancel{+t^3e^t} - (t^2e^t \cancel{+t^3e^t} \cancel{+2te^t} + 2t^2e^t) + t^2e^t \cancel{+2te^t}$

$= 2t^2e^t - t^2e^t - 2t^2e^t + t^2e^t = 0$ so te^t is a sol to homogeneous eq.

For t: $(t)'' = 0 \quad (t)' = 1$

$t^2(0) - (t^2+2t)(1) + (t+2)(t) = t^2+2t - t^2 - 2t = 0$ so t is also sol to // //

So general solution $y(t)$ is

$y(t) = y_p(t) + y_h(t)$ ← choose any y_p, I chose $y_1(t)$

$$\boxed{y(t) = e^t(2t+1) - (t+1) + C_1te^t + C_2t}$$

(Note: $\frac{te^t}{t} = e^t \neq c$ so te^t & t are lin. ind. so te^t & t form a fundamental set of solutions to the homogeneous equation, so $y_h = C_1te^t + C_2t$)

Exercise 6. (9pt) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (5pt)

$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ $\quad T = 4 \quad D = 3-(-2) = 5$

$p(\lambda) = \lambda^2 - 4\lambda + 5$

$\Rightarrow \quad \lambda = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

let $\lambda = 2+i$

$A - \lambda I = \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$

$e^{(2+i)t}\begin{pmatrix} 1-i \\ -1 \end{pmatrix} = e^{2t}(\cos t + i \sin t)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$

$= e^{2t}\left(\left(\cos t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) + i\left(\sin t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \cos t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)\right)$

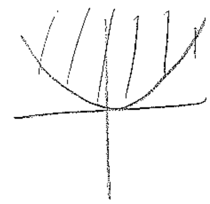
taking the real & imaginary parts to be our fundamental set of solutions:

$$\boxed{y(t) = c_1 e^{2t}\left(\cos t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) + c_2 e^{2t}\left(\sin t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \cos t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}$$

OR

$$\boxed{y(t) = c_1 e^{2t}\begin{pmatrix} \cos t + \sin t \\ -\cos t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \sin t - \cos t \\ -\sin t \end{pmatrix}}$$

$T^2-4D<0$



(2) Conclude that the equilibrium point is a spiral. (1pt)

The trace $T=4$, determinant $D=5$,

So $T^2-4D = 16-20 = -4 < 0$. Since $T \neq 0$, and $T^2-4D<0$, the equilibrium point must be a spiral.

Furthermore, if we look at our general solution, the cosine & sine functions are periodic, so they themselves trace out elliptical paths around the origin. Together with the e^{2t} terms, the graph turns into a spiral.

(3) Is it a sink or a source? (1pt)

Source.

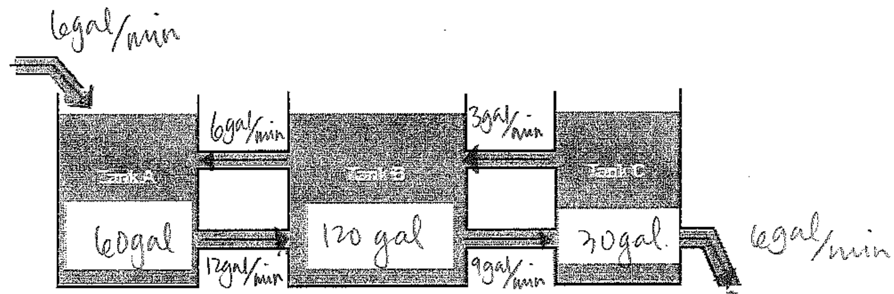
(4) Does the spiral rotate clockwise or counterclockwise? (2pt)

at $(1,0)$:

$$\vec{y}' = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \nearrow$$

Since the tangent vector at $(1,0)$ points up, the spiral rotates counterclockwise.

Prob diag 2018.pdf



Exercise 7. (9pt)
Consider the above mixing problem with the following data.

- at time $t = 0$ there is 0 lb of salt in tank A, 10 lb of salt in tank B, and 20 lb of salt in tank C.
- Tank A contains initially 60 gallons of solution, Tank B contains initially 120 gallons of solution and Tank C contains initially 30 gallons of solution.
- 6 gal/min of water enters tank A through the top far left pipe.
- The solutions flows at
  - at 6 gal/min through the upper left pipe
  - at 12 gal/min through the lower left pipe
  - at 3 gal/min through the upper right pipe
  - at 9 gal/min through the lower right pipe
- 6 gal/min of solutions leaves tank C through the bottom far right pipe.

Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ in tank A, B, and C at time t (**you do NOT have to solve it!**).

$x_A(0) = 0 \qquad x_B(0) = 10 \qquad x_c(0) = 20$

Analyze volume:

tank A: 6+6 gal/min entering, 12 gal/min leaving

tank B: 12+3 gal/min entering, 6+9 gal/min leaving

tank C: 9 gal/min entering, 3+6 gal/min leaving

volume rates balanced for all tanks, so volume constant over time.

$$x_A' = \text{rate in} - \text{rate out} = 6\left(\frac{x_B}{120}\right) + 6(0) - 12\left(\frac{x_A}{60}\right)$$

$$x_B' = \text{rate in} - \text{rate out} = 12\left(\frac{x_A}{60}\right) + 3\left(\frac{x_c}{30}\right) - (6+9)\left(\frac{x_B}{120}\right)$$

$$x_c' = \text{rate in} - \text{rate out} = 9\left(\frac{x_B}{120}\right) - (3+6)\left(\frac{x_c}{30}\right)$$

$$\vec{x}' = \begin{pmatrix} -12/60 & 6/120 & 0 \\ 12/60 & -15/120 & 3/30 \\ 0 & 9/120 & -9/30 \end{pmatrix} \vec{x}$$

$$\boxed{\begin{pmatrix} x_A'(t) \\ x_B'(t) \\ x_c'(t) \end{pmatrix} = \begin{pmatrix} -1/5 & 1/20 & 0 \\ 1/5 & -1/8 & 1/10 \\ 0 & 3/40 & -3/10 \end{pmatrix} \begin{pmatrix} x_A(t) \\ x_B(t) \\ x_c(t) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x_A(0) \\ x_B(0) \\ x_c(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}}$$

or in equation form:

$$x_A'(t) = -\tfrac{1}{5}x_A(t) + \tfrac{1}{20}x_B(t)$$

$$x_B'(t) = \tfrac{1}{5}x_A(t) - \tfrac{1}{8}x_B(t) + \tfrac{1}{10}x_c(t)$$

$$x_c'(t) = \tfrac{3}{40}x_B(t) - \tfrac{3}{10}x_c(t)$$

$x_A(0) = 0$

$x_B(0) = 10$

$x_c(0) = 20$

$(x-1)^2(x-3) = (x^2-2x+1)(x-3)$

$= x^3 - 2x^2 + x - 3x^2 + 6x - 3$

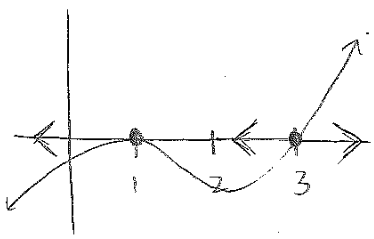
$= x^3 - 5x^2 + 7x - 3$

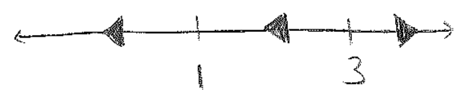
Exercise 8. (8pt)
Consider the differential equation

$$\frac{dx}{dt} = e^x(x^3 - 5x^2 + 7x - 3)$$

(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

$x' = e^x(x-1)^2(x-3)$ equilibrium points : $x = 1, 3$.

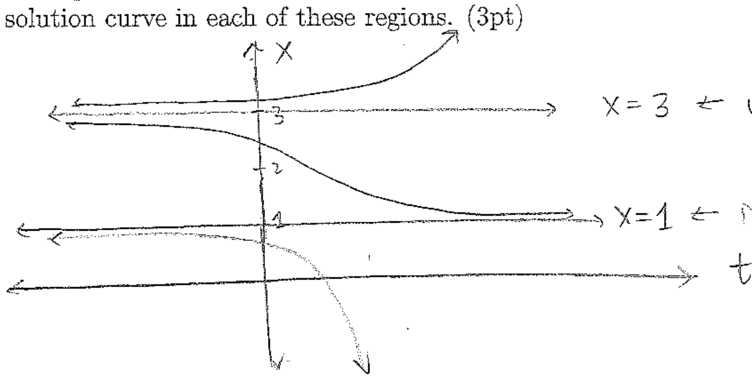




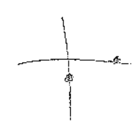
(2) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)

$x=3$ is an unstable equilibrium point

$x=1$ is an inconclusive equilibrium point.



(3) Does there exist a solution of the equation, $x(t)$, satisfying $x(0) = -1$ and $x(2) = 0$? Justify your answer. (2pt)



$x' = e^x(x^3-5x^2+7x-3) = f(x)$

$\frac{df}{dx} = e^x(x^3-5x^2+7x-3) + e^x(3x^2-10x+7)$

Both x' & $\frac{df}{dx}$ are continuous & defined for all $\mathbb{R}^2$, so the uniqueness theorem applies.

Since $x(0) = -1$, by uniqueness theorem it cannot touch $x=1$ so it will always lie below $x=1$. Thus, x' will always be negative. In other words, $x(t)$ is decreasing so if $x(0) = -1$, $x(2)$ cannot be 0.

Exercise 9. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx}$$

(1) Find all constant solutions of the above equation. (4pt)

$$x' = 0 \Rightarrow \frac{x^2-3x+2}{tx} = 0 \Rightarrow x^2-3x+2=0 \Rightarrow (x-2)(x-1)=0$$

$$\boxed{x(t) = 1 \quad \text{and} \quad x(t) = 2}$$

(2) Argue that the range of the solution to the initial value problem $x(1) = 1.2$ is contained in $(1, 2)$. (3pt)

$$x' = \frac{x^2-3x+2}{tx} = f(x)$$

$$\frac{\partial f}{\partial x} = \frac{(2x-3)(tx) - t(x^2-3x+2)}{(tx)^2}$$

Since both x' & $\frac{\partial f}{\partial x}$ are defined & continuous for $t \neq 0$ $x \neq 0$, the uniqueness theorem applies for this problem.

By uniqueness theorem, $x(t)$ cannot touch $x=1$ or $x=2$. Thus if $x(1) = 1.2$, since $1 < 1.2 < 2$, $x(t)$ must be contained within the range $(1, 2)$.

(3) Can you apply the existence theorem to the initial value problem $x(0) = 5$? (1pt) Justify your answer. (1pt)

No, because in order to apply the existence theorem, $\frac{dx}{dt}$ must be defined and continuous in a rectangle R containing that point, but at $(0,5)$, $t=0$, and $\frac{dx}{dt} = \frac{x^2-3x+2}{tx}$ is not defined at $t=0$.

Exercise 10. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$2(x + xy^2) + b(x^m y + y^2)\frac{dy}{dx} = 0$$

$$\underbrace{2(x+xy^2)}_{P}\,dx + \underbrace{b(x^m y+y^2)}_{Q}\,dy = 0$$

For the equation to be exact, $\frac{\partial P}{\partial y}$ must $= \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = 4xy \qquad \frac{\partial Q}{\partial x} = m\,bx^{m-1}y$

$\Rightarrow \quad m-1 = 1 \Rightarrow m=2, \; b=2.$

(2) Solve the equation using the value of b and m you obtained in part (a). (6pt)

$$2(x+xy^2)\,dx + 2(x^2y + y^2)\,dy = 0$$

$$F(x,y) = \int 2x + 2xy^2\,dx = x^2 + x^2y^2 + \phi(y)$$

$$2x^2y + 2y^2 = \frac{\partial(x^2+x^2y^2+\phi(y))}{\partial y} = 2x^2y + \phi'(y)$$

$$\Rightarrow \phi'(y) = 2y^2 \Rightarrow \phi(y) = \frac{2}{3}y^3$$

$$F(x,y) = \boxed{x^2 + x^2y^2 + \frac{2}{3}y^3 = C}$$

Exercise 11. (11pt) Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$

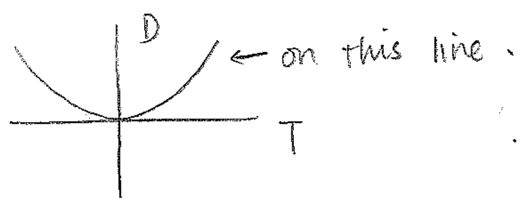
$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(1) Determine where in the trace-determinante plane the system $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$ fit. (3pt)

$A:\ T = 6 \quad D = 9 \Rightarrow T^2 - 4D = 0$

$B:\ T = 4 \quad D = 4 \Rightarrow T^2 - 4D = 0$

Both $\vec{y}' = A\vec{y}$ & $\vec{x}' = B\vec{x}$ lie on the ~~thee~~ $T^2 - 4D = 0$ line in the TD-plane



(2) Find all of the half line solutions for the system $\vec{y}' = A\vec{y}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$p(\lambda) = \lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0$

$\lambda = 3:$

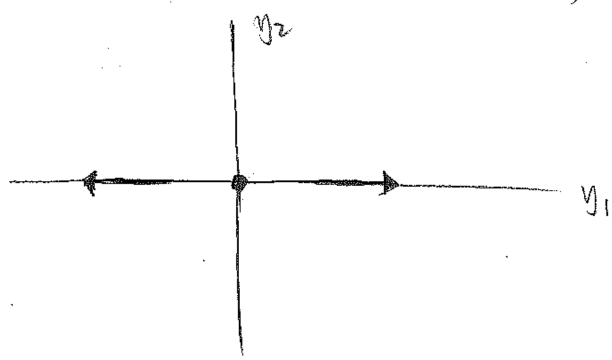
$A - 3I = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

find $\vec{w}$ s.t. $(A - 3I)\vec{w} = \vec{v} \Rightarrow \vec{w} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$

$y(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \left(\begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$

Half line solutions (2):

$y(t) = \pm c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



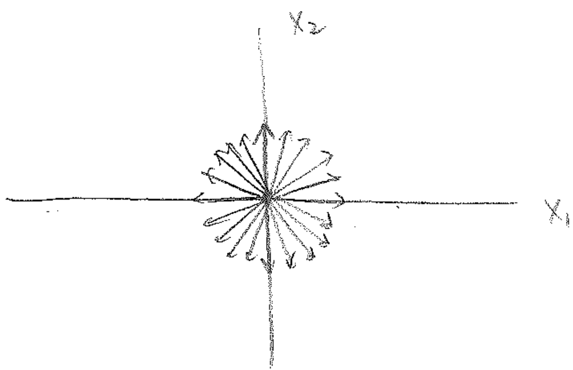
(3) Find all of the half line solutions for the system $\vec{x}' = B\vec{x}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\lambda = 2$

$B - \lambda I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

There are infinitely many half-line solutions in the form of $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$



Exercise 12. (5pt)

(1) Consider the second order equation $y'' + 3t^2y' - \cos(t)y = -3e^t$. Write this equations as a planar system of first-order equations. (2pt)

let $v = y'$

$$\boxed{\begin{aligned} y' &= v \\ v' &= -3t^2 v + \cos(t)\, y - 3e^t \end{aligned}}$$

(2) Consider more generally an n-order equation $y^{(n)} = F(t, y, \ldots, y^{(n-1)})$. How can you write this as a system of first-order equations? (3pt)

let
$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{(n-1)} \end{aligned}$$

Then:
$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n) \end{aligned}}$$

Extra page