33B Final

TOTAL POINTS

100 / 100

QUESTION 1

1 homogenous eon **7 / 7**

✓ **+ 2 pts** Homogeneous

✓ **+ 1 pts** Substitution

+ 3 pts Single-Variable Integrating Factor

✓ **+ 2 pts** Making Exact

✓ **+ 2 pts** Solving

+ 0 pts No Points

QUESTION 2

2 separable eqn **5 / 5**

✓ **- 0 pts** Correct

- 1 pts minor mistake

- 1 pts need more simplification

- 5 pts no work

- 3 pts know it's separable equation, fail to do the partial fraction decomposition

- 2 pts right hand side is polynomials of x, integration can be calculated directly.

- 3 pts idea is correct, need calculation

QUESTION 3

forcing term 10 pts

3.1 polynomial **2 / 2**

✓ **- 0 pts** Correct

- 0.5 pts b=-1

- 0.5 pts c=-1

- 0.5 pts a=2

- 2 pts wrong

3.2 sin **4 / 4**

✓ **- 0 pts** Correct

- 1 pts missing cos in the Setup/t for cos in Setup/wrong second setup

- 1 pts computational mistake

- 2 pts missing 2 in the differential

- 4 pts wrong/no answer

- 2 pts missing second step

- 2 pts computational mistake

- 0.5 pts missing t in answer

- 0.5 pts missing - in the answer

- 1 pts didn't finish

- 1 pts missing 1 step

3.3 general solution **4 / 4**

✓ **- 0 pts** Correct

- 1 pts no/wrong characteristic polynomial

- 1 pts no/wrong roots

- 1 pts no/wrong homogeneous solution

- 1 pts wrong final answer

- 4 pts wrong/no answer

- 0.5 pts - missing for polynomial

- 0.5 pts missing t for cos/wrong answer for trig part

QUESTION 4

4 system **10 / 10**

✓ **- 0 pts** Correct

- 1 pts Incorrectly identified the eigenvalues or their algebraic multiplicity.

- 2 pts Incorrectly found the eigenvectors.

- 2 pts Incorrectly found generalized eigenvectors.

- 2 pts Incorrect coefficients or powers of t or (A-L I) in general solution

- 1 pts Wrong vectors in general solution.

- 2 pts Failed to solve IVP.

- 1 pts Arithmatic error

- 1 pts Got an unsolvable system when solving IVP.

QUESTION 5

2nd linear differential equation 8 pts

5.1 verify 4 / 4

✓ - **0 pts** **Correct**

- **2 pts** incorrect calculation
- **2 pts** not finished
- **4 pts** no work
- **3 pts** some work

5.2 find general solution 4 / 4

✓ - **0 pts** **Correct**

- **4 pts** Incorrect calculation of homogeneous. For second order linear differential equation, use y_g=c1*y_h1+c2*y_h2+y_p. y1,y2,y3 can be decomposed in that way, hence we can get y_h1=y1-y2, y_h2 = y2-y3.
- **2 pts** incorrect calculation of y_h1, y_h2, but idea is correct
- **3 pts** incorrect calculation of y_h1, y_h2
- **3 pts** some work
- **1 pts** y_g=c1*y_h1+c2*y_h2+y_p
- **1 pts** no calculation detail

QUESTION 6

linear system 9 pts

6.1 find general solution 5 / 5

✓ - **0 pts** **Correct**

- **3 pts** eigenvector: solve for (A-\lambda*I)v = 0.
- **2 pts** some calculation error\no finished
- **1 pts** final answer incorrect
- **5 pts** no work
- **3 pts** calculation error, incorrect eigenvalue, eigenvector, idea is correct

6.2 spiral? 1 / 1

✓ - **0 pts** **Correct**

- **1 pts** incorrect

6.3 sink\source? 1 / 1

✓ - **0 pts** **Correct**

- **1 pts** incorrect

6.4 direction? 2 / 2

✓ - **0 pts** **Correct**

- **2 pts** wrong
- **1 pts** Somework

QUESTION 7

7 9 / 9

✓ - **0 pts** **Correct**

- **1 pts** one entry wrong
- **2 pts** two entries wrong
- **3 pts** three entries wrong
- **4 pts** 4 entries wrong
- **5 pts** 5 entries wrong
- **7 pts** all entries wrong
- **1 pts** incorrect initial value
- **2 pts** initial value missing
- **9 pts** wrong/ no answer
- **2 pts** not taking concentration

QUESTION 8

8 pts

8.1 3 / 3

✓ + **1 pts** **Correct Roots**

✓ + **2 pts** **Phase Line**

+ **0 pts** No Points

8.2 3 / 3

✓ + **1 pts** **Curves**

✓ + **1 pts** **1 Stability**

✓ + **1 pts** **3 Stability**

+ **0 pts** No Points

8.3 2 / 2

✓ + **1 pts** **Correct**

✓ + **1 pts** **Justification**

+ **0 pts** No Points

QUESTION 9

9 pts

9.1 4 / 4

✓ - **0 pts** **Correct**

- **4 pts** Didn't know to solve x^2-3x+2=0.

- **2 pts** Got the wrong roots.
- **2 pts** didn't write solutions

9.2 3 / 3

✓ **- 0 pts** Correct

- 1 pts Didn't mention uniqueness theorem.

- 2 pts Didn't say that uniqueness means solution cannot cross the solutions x=1 and x=2.

9.3 2 / 2

✓ **- 0 pts** Correct

- 1 pts Wrong answer

- 1 pts Inadaquate justification.

QUESTION 10

9 pts

10.1 3 / 3

✓ **- 0 pts** Correct

- 1 pts Didn't use the definition of exact.

- 2 pts Incorrectly solved for b and m.

- 1 pts Incorrectly solved for one of b or m.

10.2 6 / 6

✓ **- 0 pts** Correct

- 1 pts Minor Calculation error.

- 2 pts Found antiderivative, but not solution (need to set F(x,y)=C).

- 2 pts Did not use the correct algorithm to solve.

- 6 pts Wrong/Blank

- 1 pts Incorrectly solved for g'(y) or h'(x).

QUESTION 11

11 pts

11.1 3 / 3

✓ **+ 1.5 pts** **Correct for A**

✓ **+ 1.5 pts** **Correct for B**

+ 0 pts No Points

11.2 4 / 4

✓ **+ 2 pts** **Eigenvector**

✓ **+ 2 pts** **Sketch**

+ 0 pts No Points

11.3 4 / 4

✓ **+ 2 pts** **Two Eigenvectors**

✓ **+ 2 pts** **Star Behavior**

+ 0 pts No points

QUESTION 12

5 pts

12.1 2 / 2

✓ **- 0 pts** Correct

- 0.5 pts did not solve for v' (correctly)

- 2 pts no answer/ wrong answer

- 1 pts wrong substitution

12.2 3 / 3

✓ **- 0 pts** Correct

- 3 pts wrong/no answer

- 2 pts for trying

- 1 pts missing F(t, y, ... , y_{n-1}) in answer/missing ' in the answer

- 1.5 pts only using one variable

- 1 pts missing equations in answer

- 1 pts using y^{i} in equations

- 1 pts not adding additional equations

gradescope

FINAL

12/10/2018

Math33B
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Name:

UID:

section:

Problem	Points	Score
1	7	
2	5	
3	10	
4	10	
5	8	
6	9	
7	9	
8	8	
9	9	
10	9	
11	11	
12	5	
Total	100	

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Solve the following equation. (*Hint: Find the integrating factor*)

$$(x^2 + y^2)\, dx - 2xy\, dy = 0$$

This is a homogeneous equation. We let $y = vx$.

$$(x^2 + (vx)^2)\, dx - 2x(vx)(v\,dx + x\,dv) = 0.$$

$$\frac{1}{x^2} \quad (1 + v^2)\, dx - 2v(v\,dx + x\,dv) = 0$$

$$(1 + v^2 - 2v^2)\, dx - 2vx\, dv = 0$$

$$(1 - v^2)\, dx = 2vx\, dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2}\, dv.$$

$$\ln|x| = -\ln|1 - v^2| + C_0$$

$$|x| = \frac{C_1}{|1 - v^2|}$$

Plugging in $v = \frac{y}{x}$ yields

$$|x| = \frac{C_1}{|1 - (\frac{y}{x})^2|}$$

$$\boxed{x - \frac{y^2}{x} = C_1}$$ implicitly defines the solutions.

Exercise 2. (5pt) Solve $y' = y(y+1)(x+2)(x+3)$

$$\frac{dy}{dx} = y(y+1)(x+2)(x+3)$$

$$\int \frac{dy}{y(y+1)} = \int (x+2)(x+3)\,dx.$$

$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \int (x^2+5x+6)\,dx$$

$$\ln|y| - \ln|y+1| = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C_0$$

$$\ln\left|\frac{y}{y+1}\right| = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C_0.$$

$$\frac{y}{y+1} = -C_1 e^{x^3/3 + \frac{5}{2}x^2 + 6x} \qquad C_1 > 0.$$

$$\frac{1}{y+1} = 1 - C_1 e^{x^3/3 + 5/2x^2 + 6x}.$$

$$y+1 = \frac{1}{1 - C_1 e^{x^3/3 + 5/2x^2 + 6x}}$$

$$\boxed{y = \frac{1}{1 - C_1 e^{x^3/3 + (5/2)x^2 + 6x}} - 1}$$

Exercise 3. (10pt) Find a **particular** solution to the following two differential equations

(1) $y'' + 4y = 8t^2 - 4t$ (2pt)

Guess $y_p = at^2 + bt + c.$

$y_p' = 2at + b$

$y_p'' = 2a.$

$$\boxed{y_p(t) = 2t^2 - t - 1.}$$

$2a + 4at^2 + 4bt + 4c = 8t^2 - 4t.$

$4a = 8 \Rightarrow a = 2$

$4b = -4 \Rightarrow b = -1$

$2a + 4c = 0 \Rightarrow c = -1$

(2) $y'' + 4y = 4\sin(2t)$ (4pt)

Guess $y_p(t) = a \cdot \sin(2t) + b\cos(2t).$

$y_p''(t) = -4a\sin(2t) - 4b\cos(2t).$

$y_p'' + 4y_p = 0 \rightarrow$ degenerate.

Guess $y_p(t) = at\sin(2t) + bt\cos(2t).$

$y_p'(t) = 2t(a\cos(2t) - b\sin(2t)) + a\sin(2t) + b\cos(2t)$

$y_p''(t) = -4t(a\sin(2t) + b\cos(2t)) + 2a\cos(2t) - 2b\sin(2t)$
$+ 2a\cos(2t) - 2b\sin(2t)$

$= -4t(a\sin(2t) + b\cos(2t)) + 4a\cos(2t) - 4b\sin(2t).$

$y_p'' + 4y_p = 4a\cos(2t) - 4b\sin(2t) = 4\sin(2t).$

$\Downarrow$

$4a = 0 \Rightarrow a = 0$

$-4b = 4 \Rightarrow b = -1.$

$$\boxed{y_p(t) = -t\cos(2t).}$$

$y'' + 4y = ate^{2t}$
$y = ate^{2t}$

Check: $y_p' = 2t\sin(2t) - \cos(2t)$

$y_p'' = 4t\cos(2t) + 2\sin(2t) + 2\sin(2t)$

$= 4t\cos(2t) + 4\sin(2t).$ ✓

(3) Give the **general** solution to the following differential equation

$$y'' + 4y = 8\sin(2t) - 8t^2 + 4t. \text{ (4pt)}$$

We ~~consider~~ consider the homogeneous equation.

$$y'' + 4y = 0.$$

$$\lambda^2 + 4 = 0.$$

$$\lambda = 2i, \quad \bar{\lambda} = -2i.$$

$$e^{2it} = \cos(2t) + i\sin(2t).$$

$$y_h(t) = C_1\cos(2t) + C_2\sin(2t).$$

We then find a particular solution of the inhomogeneous ~~solution~~ equation using a linear combination of (1) and (2).

$$8\sin(2t) - 8t^2 + 4t = 2(4\sin(2t)) - (8t^2 - 4t).$$

$$y_p(t) = -2t\cos(2t) - (2t^2 - t - 1).$$

~~Th~~ We finally arrive at the general solution:

$$\boxed{y(t) = C_1\cos(2t) + C_2\sin(2t) - 2t\cos(2t) - (2t^2 - t - 1).}$$

Exercise 4. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

TODO

$$\vec{y}' = \underbrace{\begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{A} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 1 \\ -6 \\ -2 \end{pmatrix}$$

$\lambda_1 = -1$
$\mathrm{almu}(\lambda_1) = 3$
$\mathrm{gemu}(\lambda_1) = ?$

$\lambda_2 = 3$
$\mathrm{almu}(\lambda_2) = 1.$
$\mathrm{gemu}(\lambda_2) = 1.$

$$A - \lambda_2 I = \begin{bmatrix} -4 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 \end{bmatrix}. \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} \in \ker(A - \lambda_2 I)$$

$$E_{\lambda_1} = A - \lambda_1 I = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \dim \ker(A - \lambda_1 I) = 1. \Rightarrow \mathrm{gemu}(\lambda_1) = 1.$$

$$(A - \lambda_1 I)^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \dim \ker(A - \lambda_1 I)^2 = 2 < \mathrm{almu}(\lambda_1)$$

$$(A - \lambda_1 I)^3 = \begin{bmatrix} 0 & 0 & 8 & -6 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 64 & -48 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \dim \ker(A - \lambda_1 I)^3 = 3 = \mathrm{almu}(\lambda_1).$$

We pick $\vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \in \ker(A - \lambda_1 I)^3 \setminus \ker(A - \lambda_1 I)^2 \setminus \ker(A - \lambda_1 I).$

$$\vec{w}_2 = (A - \lambda_1 I)\vec{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_3 = (A - \lambda_1 I)^2 \vec{w}_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{y}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} + e^{-t}\left((C_2 + C_3 t + C_4 \tfrac{t^2}{2}) \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (C_3 + C_4 t) \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + C_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \right)}$$

$-9+57$
$-9 \cdot ... = 48$
$-\frac{9}{2} - \frac{3}{2}$

$$\vec{y}(0) = C_1 \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + C_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -6 \\ -2 \end{bmatrix}.$$

$4C_4 = -2 \Rightarrow C_4 = -\frac{1}{2}.$

$8C_1 + 3C_4 = -6$

$8C_1 = -\frac{9}{2} \Rightarrow C_1 = -\frac{9}{16}$

$-\frac{9}{16} + 6C_2 = 3$

$6C_2 = \frac{57}{16}$

$C_2 = \frac{19}{32}$

$-\frac{9}{8} + 3C_3 = 1$

$3C_3 = \frac{17}{8}. \quad C_3 = \frac{17}{24}.$

$$\vec{y}_p(t) = -\frac{9}{16} e^{3t} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} + e^{-t}\left((\tfrac{19}{32} + \tfrac{17}{24}t - \tfrac{t^2}{4}) \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (\tfrac{17}{24} - \tfrac{1}{2}t) \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \right)$$

Exercise 5. (8pt) Consider the differential equation

$$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t-1) - t(e^t+1), \qquad (t>0)$$

(1) Show that $y_1 = e^t(2t+1) - (t+1)$ is solutions to the above equation. (4pt)
(Show ALL your calculations in detail for full credit)

$y_1'(t) = 2(te^t + e^t) + e^t - 1 = 2te^t + 3e^t - 1.$

$y_1''(t) = 2(te^t + e^t) + 3e^t = 2te^t + 5e^t.$

$LHS = t^2(2te^t + 5e^t) - (t^2+2t)(2te^t + 3e^t - 1) + (t+2)(2te^t + e^t - t - 1)$

$= \cancel{2t^3e^t} + \cancel{5t^2e^t} - \cancel{2t^3e^t} - \cancel{3t^2e^t} + \cancel{t^2} - \cancel{4t^2e^t} - 6te^t + \cancel{2t} + \cancel{2t^2e^t} + te^t - \cancel{t^2} - t + 4te^t + 2e^t - \cancel{2t} - 2$

$= -te^t - t + 2e^t - 2$

$= 2(e^t - 1) - t(e^t+1)$ ✓

(2) Given that $y_2 = e^t(t+1)+(t-1)$, and $y_3 = e^t(1-t)+(2t-1)$ are also solutions to the above equation, find the general solution to the equation. Justify your answer. (4pt)

Given that this is an inhomogeneous second-order differential equation, we ought to find two linearly independent solutions to the ass. homogeneous equation.

$y_{h1} = y_1 - y_2 = te^t - 2t.$

$y_{h2} = y_2 - y_3 = 2te^t - t.$

Check linear independence.

$$W_{y_{h1}, y_{h2}} = \begin{vmatrix} te^t - 2t & 2te^t - t \\ te^t + e^t - 2 & 2te^t + 2e^t - 1 \end{vmatrix}$$

$= \cancel{\dots} - 4t^2e^t - 4e^t + \cancel{2t}$
$- \cancel{2t^2e^{2t}} - 2te^{2t} + 4te^t \dots$
$- \cancel{2t}$ $= t^2e^t - 4e^t \neq 0.$ ✓

Since the solution to an inhomogeneous second-order equation is $y(t) = y_h(t) + y_p(t)$, where y_h is the general solution to the associated homogeneous equation, and y_p is a particular solution to the IHE, we have

$$y(t) = \underbrace{C_1(te^t - 2t) + C_2(2te^t - t)}_{y_h} + \underbrace{e^t(t+1) + (t-1)}_{y_p = y_2}$$

Check: $y'_{h1} = (te^t + e^t - 2)(t^2+2t) = \cancel{t^3e^t} - \cancel{t^2e^t} + \cancel{2t^2} - 2t^2e^t - \cancel{2te^t} + \cancel{4t}$

$y''_{h1} = (te^t + 2e^t)t^2 = \cancel{t^3e^t} + \cancel{2t^2e^t}$

$(te^t - 2t)(t+2) = \cancel{t^2e^t} + \cancel{2te^t} - \cancel{2t^2} - \cancel{4t}$

Exercise 6. (9pt) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (5pt)

We find eigenvalues/vectors of A.

$p(\lambda) = \lambda^2 - 4\lambda + 5 = 0$

$$\lambda = 2 + i. \quad \bar{\lambda} = 2 - i.$$

$$E_\lambda = \ker \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} = \operatorname{span} \left(\begin{bmatrix} 1-i \\ -1 \end{bmatrix} \right).$$

$$\vec{y}_p(t) = e^{(2+i)t} \begin{bmatrix} 1-i \\ -1 \end{bmatrix} = e^{2t} \left(e^{it} \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \right)$$

$$= e^{2t} \left(\begin{bmatrix} \cos(t) + \sin(t) \\ -\cos(t) \end{bmatrix} + i \begin{bmatrix} \sin(t) - \cos(t) \\ -\sin(t) \end{bmatrix} \right).$$

$$\vec{y}(t) = e^{2t} \left(C_1 \begin{bmatrix} \cos(t) + \sin(t) \\ -\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) - \cos(t) \\ -\sin(t) \end{bmatrix} \right).$$

(2) Conclude that the equilibrium point is a spiral. (1pt)

Two complex eigenvalues $\Longleftrightarrow$ $T^2 < 4D$

$$\Downarrow \qquad \underset{16}{\overset{\shortparallel}{T^2}} < \underset{20}{\overset{\shortparallel}{4D}}$$

spiral

(3) Is it a sink or a source? (1pt)

$\mathrm{Re}(\lambda) > 0 \quad \Longleftrightarrow 16 = T > 0$

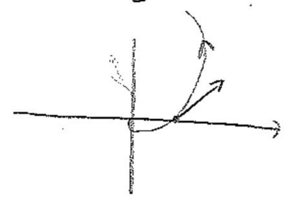
$\Downarrow$

source

(e^{2t} grows lager as t increases).

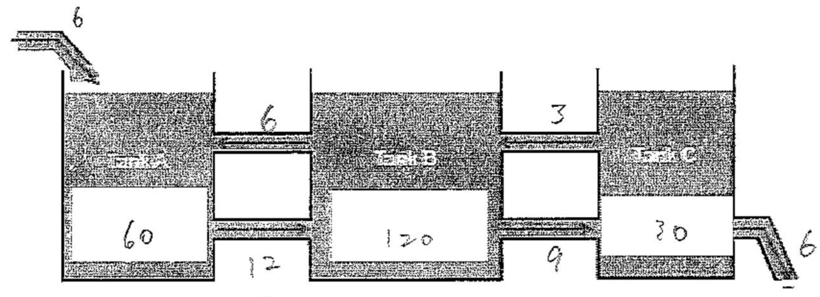
(4) Does the spiral rotate clockwise or counterclockwise? (2pt)

At $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $\vec{y}' = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



counterclockwise.

Prob diag 2018.pdf



Exercise 7. (9pt)
Consider the above mixing problem with the following data.

- at time $t = 0$ there is 0 lb of salt in tank A, 10 lb of salt in tank B, and 20 lb of salt in tank C.
- Tank A contains initially 60 gallons of solution, Tank B contains initially 120 gallons of solution and Tank C contains initially 30 gallons of solution.
- 6 gal/min of water enters tank A through the top far left pipe.
- The solutions flows at
  - at 6 gal/min through the upper left pipe
  - at 12 gal/min through the lower left pipe
  - at 3 gal/min through the upper right pipe
  - at 9 gal/min through the lower right pipe
- 6 gal/min of solutions leaves tank C through the bottom far right pipe.

Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ in tank A, B, and C at time t (**you do NOT have to solve it!**).

~~Let $v_A(t)$, $v_B(t)$, $v_C(t)$ represent volume~~

We see that volume in each tank stays constant.

$x_A(0) = 0$.
$x_B(0) = 10$.
$x_C(0) = 20$.

$$x_A'(t) = -\frac{12}{60}x_A(t) + \frac{6}{120}x_B(t)$$

$$x_B'(t) = +\frac{12}{60}x_A(t) - \frac{15}{120}x_B(t) + \frac{3}{30}x_C(t)$$

$$x_C'(t) = +\frac{9}{120}x_B(t) - \frac{9}{30}x_C(t)$$

$(x^2-2x+1)(x-3)$
$= (x^3-5x^2+7x-3)$

Exercise 8. (8pt)
Consider the differential equation

$$\frac{dx}{dt} = e^x(x^3 - 5x^2 + 7x - 3)$$

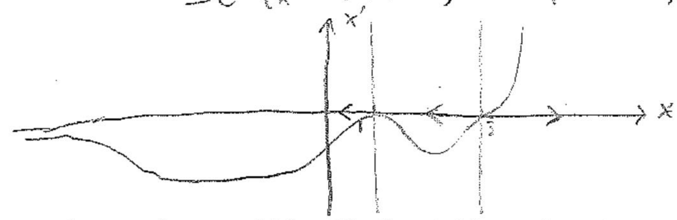
(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

We observe $\left.\frac{dx}{dt}\right|_{x=1} = e(1-5+7-3) = 0$. $\therefore (x-1)$ is a factor of (x^3-5x^2+7x-3)

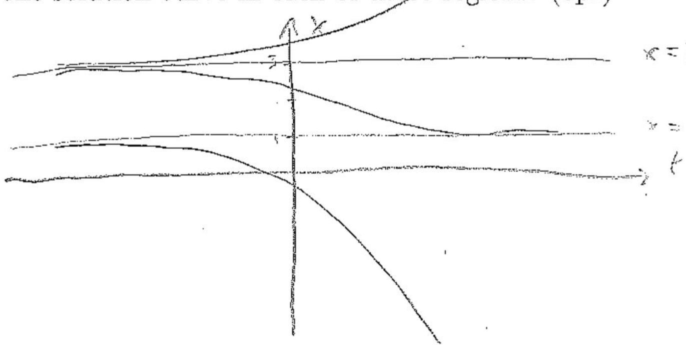
$$\begin{array}{r} x^2 - 4x + 3 \\ x-1 \,\big)\, \overline{x^3 - 5x^2 + 7x - 3} \\ x^3 - x^2 \quad\quad\quad \\ \hline -4x^2 + 7x \quad \\ -4x^2 + 4x \quad \\ \hline 3x - 3 \\ 3x - 3 \\ \hline 0 \end{array}$$

$\frac{dx}{dt} = e^x(x-1)(x^2-4x+3)$
$= e^x(x-1)^2(x-3)$

equilibrium points : $\frac{dx}{dt} = 0$
$x = 1$
$x = 3$



(2) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)



x=3 unstable
x=1 no conclusion / semi-stable

(3) Does there exist a solution of the equation, $x(t)$, satisfying $x(0) = -1$ and $x(2) = 0$? Justify your answer. (2pt)

No. (1) We see that $\frac{dx}{dt} < 0$ is strictly true for all $x < 1$. (2) By the Existence and ~~Uniqueness~~ theorem, as both $\frac{dx}{dt}$ and $\frac{d}{dx}\left(\frac{dx}{dt}\right)$ are continuous everywhere, if a solution that satisfies $x(0) = -1$ and $x(2) = 0$ were to exist, it must be continuous everywhere.

From (1), we see that there exists no continuous solution that can increase as t goes from 0 to 2; from (2), we see that no ~~such~~ solutions exist at all.

Exercise 9. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx}$$

(1) Find all constant solutions of the above equation. (4pt)

$$\frac{dx}{dt} = 0 = \frac{(x-2)(x-1)}{tx}$$

$$\boxed{\begin{array}{l} x(t) = 2 \\ x(t) = 1 \end{array}}$$

(2) Argue that the range of the solution to the initial value problem $x(1) = 1.2$ is contained in $(1, 2)$. (3pt)

Both $\frac{dx}{dt}$ and $\frac{d}{dx}\left(\frac{dx}{dt}\right) = \frac{2x-3}{tx} - \frac{x^2-3x+2}{tx^2}$ are continuous for all $x > 0$, and therefore uniqueness applies to solutions in that range. Since $x(t) = 2$ and $x(t) = 1$ are solutions, any solution that satisfies $x(1) = 1.2$, 1) must be continuous, and 2) must not cross $x = 1$ or $x = 2$. As such, the given statement is true, and such a solution is contained in $(1,2)$.

(3) Can you apply the existence theorem to the initial value problem $x(0) = 5$? (1pt) Justify your answer. (1pt)

No. $\frac{dx}{dt}$ is not defined (and therefore not continuous) at $t = 0$ so the existence theorem does not apply at that point.

Exercise 10. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$2(x + xy^2) + b(x^m y + y^2)\frac{dy}{dx} = 0$$

$$2(x + xy^2)dx + b(x^m y + y^2)dy = 0.$$

$$\frac{\partial}{\partial y} 2(x + xy^2) = \frac{\partial}{\partial x} b(x^m y + y^2)$$

$$4xy = bmx^{m-1} y.$$

$$m - 1 = 1 \Rightarrow \boxed{m = 2}$$

$$bm = 4 \Rightarrow \boxed{b = 2}$$

(2) Solve the equation using the value of b and m you obtained in part (a). (6pt)

$$2(x + xy^2)dx + 2(x^2 y + y^2)dy = 0$$

$$F(x,y) = \int 2(x + xy^2)dx + g(y) = x^2 + x^2y^2 + g(y).$$

$$\frac{\partial F}{\partial y} = 2x^2y + g'(y) = 2x^2y + 2y^2$$

$$g'(y) = 2y^2$$

$$g(y) = \frac{2}{3}y^3.$$

$F(x,y) = \boxed{x^2 + x^2y^2 + \frac{2}{3}y^3 = C}$ implicitly defines the solutions.

Exercise 11. (11pt) Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

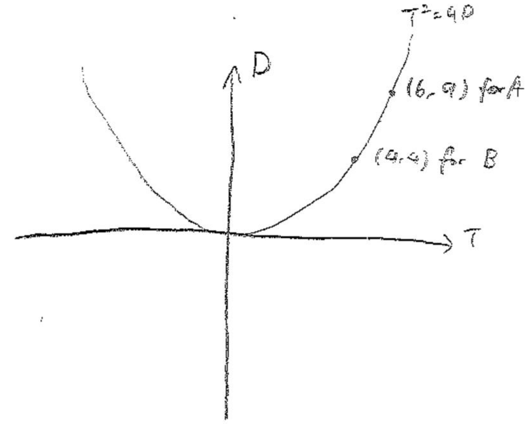
(1) Determine where in the trace-determinante plane the system $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$ fit. (3pt)

For $\vec{y}' = A\vec{y}$: $T = 6, \rightarrow$ source
$D = 9$, $\quad T^2 = 4D = 36$: degenerate node

$\boxed{\text{degenerate nodal source}}$

For $\vec{x}' = B\vec{x}$: $T = 4 \rightarrow$ source
$D = 4$. $\quad T^2 = 4D = 16$

$\boxed{\text{degenerate nodal source}}$



(2) Find all of the half line solutions for the system $\vec{y}' = A\vec{y}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

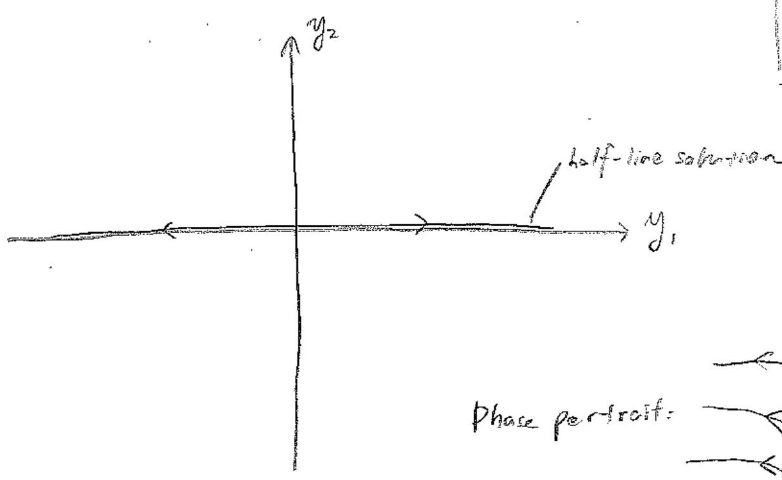
$(A-\lambda I)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\lambda = 3$
algm$(\lambda) = 2$.

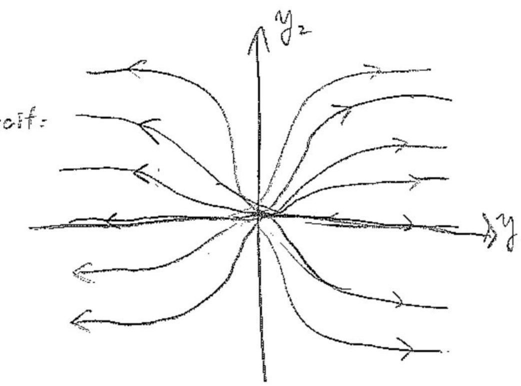
$A - \lambda I = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\boxed{\vec{y} = Ce^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

general solution: $\boxed{\vec{y} = e^{3t}\left(C_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (C_1 t + C_2)\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right)}$



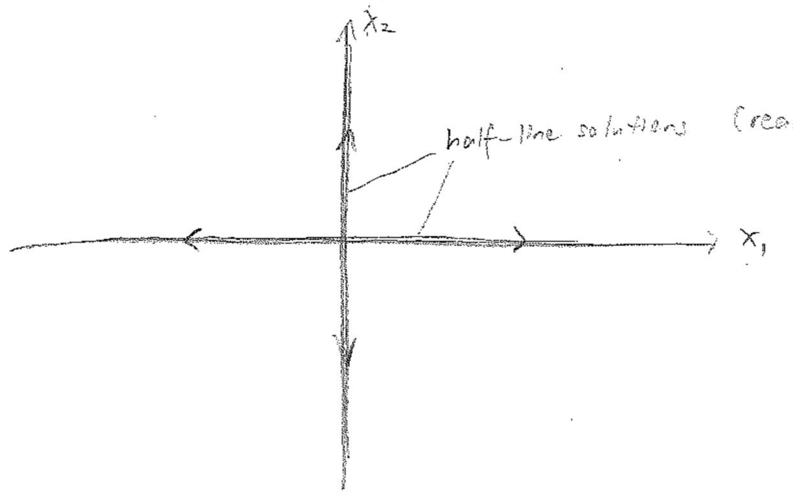
Phase portrait:



(3) Find all of the half line solutions for the system $\vec{x}' = B\vec{x}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

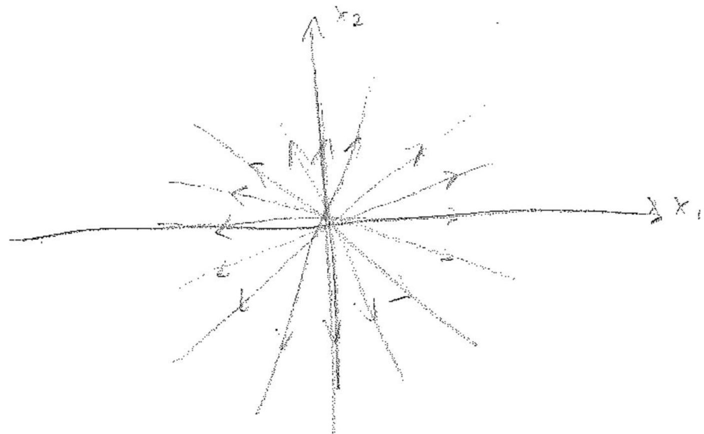
$\lambda = 2$. $\quad A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\vec{y} = e^{2t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{2t} \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$.



(really can be any two distinct lines that cross the origin)

Phase portrait:



Exercise 12. (5pt)

(1) Consider the second order equation $y'' + 3t^2y' - \cos(t)y = -3e^t$. Write this equations as a planar system of first-order equations. (2pt)

Let $v = y'$.

$$\begin{cases} v' = (-3t^2)v + (\cos t)y - 3e^t \\ y' = v \end{cases}$$

(2) Consider more generally an n-order equation $y^{(n)} = F(t, y, \ldots, y^{(n-1)})$. How can you write this as a system of first-order equations? (3pt)

Let $v_i = y^{(i)}$.

$$\begin{cases} y' = v_1 \\ v_1' = v_2 \\ \vdots \\ v_{n-2}' = v_{n-1} \\ v_{n-1}' = F(t, y, v_1, \ldots, v_{n-1}) \end{cases}$$