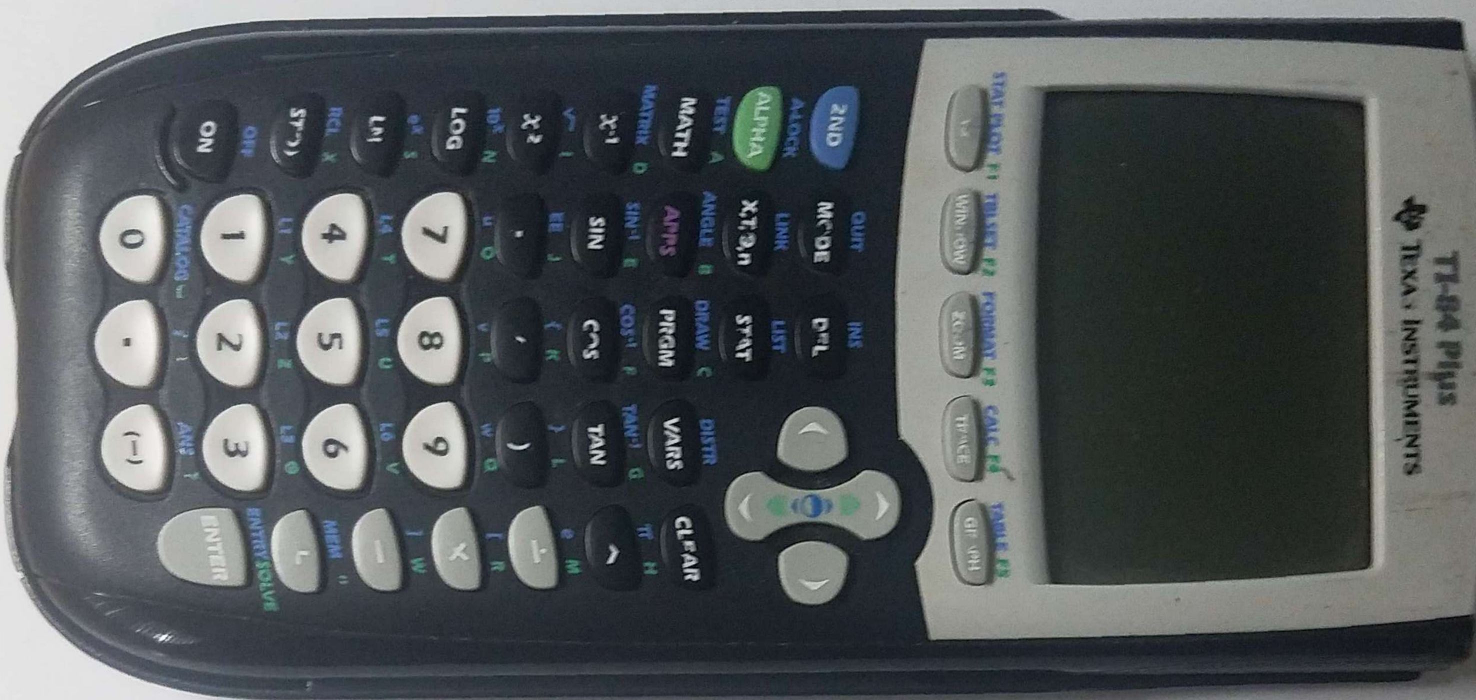
Midterm 2



Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators, books, notes, or any other material to help you.** Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	12	12
2	12	12
3	13	13
4	13	13
Total:	50	50

$y = t^{-2}$ $\quad y' = -2t^{-3}$ $\quad y'' = 6t^{-4}$

$6t^{-2} - 8t^{-2} + 2t^{2} = 0$ $\quad \checkmark$

$\checkmark$ 1. Consider the differential equation

$$t^2 y'' + 4ty' + 2y = 0 \tag{$\dagger$}$$

(a) (2 points) Verify that $y_1(t) = \frac{1}{t}$ is a solution.

$y_1'(t) = -t^{-2}$ $\quad y_1''(t) = 2t^{-3}$

$t^2(2t^{-3}) + 4t(-t^{-2}) + 2t^{-1}$

$= 2t^{-1} - 4t^{-1} + 2t^{-1} = 0$ $\quad \checkmark \quad \checkmark$

(b) (5 points) Suppose $y_2(t) = v(t)y_1(t)$ is another solution to the differential equation above. Use this to derive a second-order, linear differential equation for $v(t)$

$y_2 = vt^{-1}$

$y_2' = v' \cdot t^{-1} + v \cdot -t^{-2}$

$y_2'' = v''t^{-1} + v' \cdot -t^{-2} + v' \cdot -t^{-2} + 2vt^{-3}$

$= v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

$t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 4t(v't^{-1} + v \cdot -t^{-2}) + 2vt^{-1}$

$= v''t - 2v' + \cancel{2vt^{-1}} + 4v' - \cancel{4vt^{-1}} + \cancel{2vt^{-1}} = 0$

$= \boxed{v''t + 2v' = 0}$ $\quad \boxed{v'' + 2t^{-1}v' = 0}$ $\quad \checkmark$

(c) (5 points) Solve the differential equation from part (b) and state the general solution to ($\dagger$).

$v''t + 2v' = 0$

$v'' = \frac{-2v'}{t}$

$v' = Ae^{\int -\frac{2}{t}dt} = Ae^{-2\ln t} = At^{-2}$

$\boxed{v(t) = \int At^{-2}\,dt = -At^{-1} + B}$

$v(t) \cdot t^{-1} = at^{-2} + bt^{-1}$ $\quad \checkmark$

$v(t) = at^{-1} + b$

$\boxed{\text{general solution: } y(t) = at^{-2} + bt^{-2} + c_1t^{-1} = C_1t^{-1} + C_2t^{-2}}$

$y = \cos 2t - 3\sin 2t$
$y' = -2\sin 2t - 6\cos 2t$
$y'' = -4\cos 2t + 12\sin 2t$
$-4\cos 2t + \cancel{12\sin 2t} + \cancel{12\sin 2t} + 36\cos 2t + 8\cos 2t - \cancel{24\sin 2t} = 40\cos$

2. Consider the differential equation

$$y'' - 6y' + 8y = 40\cos(2t)$$

(a) (5 points) Find a fundamental set of solutions to the associated homogeneous equation.

5

$y'' - 6y' + 8y = 0$

$s^2 - 6s + 8 = (s-4)(s-2) = 0 \quad s = 4, 2$

$$\boxed{y_h = C_1 e^{4t} + C_2 e^{2t}}$$

(b) (7 points) Find a particular solution to the inhomogeneous equation and then state the general solution.

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$z'' - 6z' + 8z = 40e^{2it}$

$z = ae^{2it}$

$z' = 2aie^{2it}$

$z'' = -4ae^{2it}$

$-4ae^{2it} - 12aie^{2it} + 8ae^{2it} = 40e^{2it}$

$4a - 12ai = 40 \qquad a = \frac{40}{4-12i} \cdot \frac{4+12i}{4+12i} = \frac{160 + 3\cdot 160i}{160} = 1+3i$

$z = (1+3i)e^{2it} = (1+3i)(\cos 2t + i\sin 2t)$

$= \cos(2t) + i\sin(2t) + 3i\cos(2t) - 3\sin 2t$

$$\boxed{y_p = \text{real}(z) = \cos 2t - 3\sin 2t}$$

$$\boxed{y = \cos 2t - 3\sin 2t + C_1 e^{4t} + C_2 e^{2t}}$$

3. Consider the differential equation

$$y'' - 8y' + 17y = 0$$

(a) (4 points) Determine the roots of the characteristic polynomial and state the associated (complex) solutions z_1, z_2.

$s^2 - 8s + 17 = 0 \quad s = \frac{8 \pm \sqrt{64 - 68}}{2} = 4 \pm i$

$z_1 = C_1 e^{(4+i)t} \quad z_2 = C_2 e^{(4-i)t}$

(b) (5 points) Explain why the real-valued functions $y_1(t) = \mathrm{Re}(z_1) = \frac{1}{2}(z_1 + z_2)$ and $y_2(t) = \mathrm{Im}(z_1) = \frac{1}{2i}(z_1 - z_2)$ are also solutions. Calculate $y_1(t)$ and $y_2(t)$ explicitly.

diff coefficients:

$z_1 = e^{4t} e^{it} = e^{4t}(\cos t + i \sin t)$

$z_2 = e^{4t} e^{-it} = e^{4t}(\cos t - i \sin t)$

also solutions:

$\frac{1}{2} z_1 + \frac{1}{2} z_2 = e^{4t} \cos t = y_1$

$\frac{1}{2i}(z_1 - z_2) = e^{4t} \sin t = y_2$

y_1 and y_2 are solutions because the sum of two solutions to a diff eq that is homogeneous is also a solution, and $\frac{y_2}{y_1} = \tan(t)$ is not constant, so y_1 and y_2 form a fundamental set of solutions to the diff eq, so $\boxed{y_1 = C_1 e^{4t} \cos t \text{ and } y_2 = C_2 e^{4t} \sin t}$

(c) (4 points) Determine the (real) solution with initial conditions $y(0) = 4$ and $y'(0) = -1$.

$y(t) = e^{4t}(C_1 \cos t + C_2 \sin t)$

$y'(t) = 4e^{4t}(C_1 \cos t + C_2 \sin t) + e^{4t}(-C_1 \sin t + C_2 \cos t)$

$y(0) = C_1 = 4$

$y'(0) = 4(4) + C_2 = -1 \quad C_2 = -17$

$\boxed{y(t) = e^{4t}(4 \cos t - 17 \sin t)}$

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$y = -\frac{t^{-1}}{4}$ $y' = \frac{t^{-2}}{4}$ $y'' = -\frac{t^{-3}}{2}$

$-\frac{t^{-1}}{2} + \frac{3t^{-1}}{4} + \frac{3t^{-1}}{4} = t^{-1}$

4. (13 points) Given that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0$$

Use variation of parameters to find the general solution to the inhomogeneous equation

$$t^2y'' + 3ty' - 3y = \frac{1}{t}$$

of the form $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$. (Remember: when calculating $y_p'(t)$, we set $v_1'y_1 + v_2'y_2 = 0$).

$y'' + 3t^{-1}y' - 3t^{-2}y = t^{-3}$

$v_1' t + v_2' t^{-3} = 0$ $v_1' + v_2' t^{-4} = 0$

$v_1' - 3v_2' t^{-4} = t^{-3}$ $-v_2' t^{-4} - 3v_2' t^{-4} = t^{-3}$

$-4v_2' t^{-4} = t^{-3}$

$v_2' = -\frac{t}{4}$

$v_1' = \frac{t^{-3}}{4}$

$v_1 = \int \frac{t^{-3}}{4}\,dt = -\frac{t^{-2}}{8}$

$v_2 = \int -\frac{t}{4}\,dt = -\frac{t^2}{8}$

13/13

$\boxed{y_p = v_1 t + v_2 t^{-3} = -\frac{t^{-1}}{8} - \frac{t^{-1}}{8} = -\frac{t^{-1}}{4}}$

$\boxed{y(t) = -\frac{t^{-1}}{4} + C_1 t + C_2 t^{-3}}$