

1. Consider the differential equation

$$t^2 y'' + 4ty' + 2y = 0 \quad (†)$$

(a) (2 points) Verify that  $y_1(t) = \frac{1}{t}$  is a solution.

$$\begin{aligned} y_1(t) &= t^{-1} & t^2 y_1'' + 4t y_1' + 2y_1 \\ y_1'(t) &= -t^{-2} & = t^2(2t^{-3}) + 4t(-t^{-2}) + 2t^{-1} \\ y_1''(t) &= 2t^{-3} & = t^{-1}(2 - 4 + 2) = 0 \quad \checkmark \end{aligned}$$

(b) (5 points) Suppose  $y_2(t) = v(t)y_1(t)$  is another solution to the differential equation above. Use this to derive a second-order, linear differential equation for  $v(t)$

$$\begin{aligned} y_2(t) &= v(t)y_1(t) = vt^{-1} \\ y_2'(t) &= v'(t)y_1(t) + v(t)y_1'(t) = v't^{-1} - vt^{-2} \\ y_2''(t) &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) = v''t^{-1} - 2v't^{-2} + 2vt^{-3} \\ t^2 y_2'' - 4t y_2' + 2y_2 &= 0 \\ t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) - 4t(v't^{-1} - vt^{-2}) + 2(vt^{-1}) &= 0 \\ -v''t - 2v' + 2vt^{-1} + 4v' - 4vt^{-1} + 2vt^{-1} &= 0 \end{aligned}$$

$$\boxed{v''t + 2v' = 0}$$

(c) (5 points) Solve the differential equation from part (b) and state the general solution to (†).

$$\begin{aligned} v''t + 2v' &= 0 \\ v''t &= -2v' \\ \frac{v''}{v'} &= -\frac{2}{t} \end{aligned}$$

$$\ln(v') = -2 \ln t$$

$$v' = t^{-2}$$

$$v = \int \frac{1}{t^2} dt$$

$$v = -\frac{1}{t}$$

$$y_2(t) = v y_1 = -\frac{1}{t} \cdot \frac{1}{t} = -t^{-2}$$

$$\boxed{y = C_1 t^{-1} + C_2 t^{-2}} \quad \checkmark$$

2. Consider the differential equation

$$y'' - 6y' + 8y = 40 \cos(2t)$$

(a) (5 points) Find a fundamental set of solutions to the associated homogeneous equation.

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$$y(t) = e^{\lambda t} \quad y'(t) = \lambda e^{\lambda t} \quad y''(t) = \lambda^2 e^{\lambda t}$$

$$e^{\lambda t} (\lambda^2 - 6\lambda + 8) = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$$\lambda = 2 \quad \lambda = 4$$

$$y_h = c_1 e^{2t} + c_2 e^{4t}$$

(b) (7 points) Find a particular solution to the inhomogeneous equation and then state the general solution.

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$$y_p(t) = a \cos(2t) + b \sin(2t)$$

$$y_p'(t) = 2(-a \sin(2t) + b \cos(2t))$$

$$y_p''(t) = 4(-a \cos(2t) - b \sin(2t))$$

$$y_p'' - 6y_p' + 8y_p = 4(-a \cos(2t) - b \sin(2t)) - 12(-a \sin(2t) + b \cos(2t)) + 8(a \cos(2t) + b \sin(2t))$$

$$= \cos(2t)(-4a - 12b + 8a) + \sin(2t)(-4b + 12a + 8b)$$

$$= \cos(2t)(4a - 12b) + (4b + 12a) \sin(2t) = 40 \cos(2t)$$

$$4a - 12b = 40$$

$$4b + 12a = 0$$

$$12a = -4b$$

$$a = -\frac{1}{3}b$$

$$-\frac{1}{3}b - 12b = 40$$

$$-\frac{40}{3}b = 40$$

$$b = -3 \quad a = 1$$

$$y_p = \cos(2t) - 3 \sin(2t)$$

$$y = c_1 e^{2t} + c_2 e^{4t} + \cos(2t) - 3 \sin(2t)$$

3. Consider the differential equation

$$y'' - 8y' + 17y = 0$$

- ✓ (a) (4 points) Determine the roots of the characteristic polynomial and state the associated (complex) solutions  $z_1, z_2$ .

$$y(t) = e^{\lambda t} \quad y'(t) = \lambda e^{\lambda t} \quad y''(t) = \lambda^2 e^{\lambda t}$$

$$e^{\lambda t} (\lambda^2 - 8\lambda + 17) = 0$$

$$\lambda = \frac{8 \pm \sqrt{64 - 68}}{2} = 4 \pm i$$

$$z_1(t) = e^{4t} (\cos(t) + i \sin(t))$$

$$z_2(t) = e^{4t} (\cos(t) - i \sin(t))$$

- (b) (5 points) Explain why the real-valued functions  $y_1(t) = \operatorname{Re}(z_1) = \frac{1}{2}(z_1 + z_2)$  and  $y_2(t) = \operatorname{Im}(z_1) = \frac{1}{2i}(z_1 - z_2)$  are also solutions. Calculate  $y_1(t)$  and  $y_2(t)$  explicitly.

$$y_1(t) = \frac{1}{2}(z_1 + z_2) = \frac{1}{2} e^{4t} (\cos(t) + i \sin(t) + \cos(t) - i \sin(t)) = \frac{1}{2} e^{4t} (2 \cos t) = e^{4t} \cos(t)$$

$$y_2(t) = \frac{1}{2i}(z_1 - z_2) = \frac{1}{2i} e^{4t} (\cos(t) + i \sin(t) - \cos(t) + i \sin(t)) = \frac{1}{2i} e^{4t} (2i \sin t) = e^{4t} \sin t$$

Since  $z_1$  and  $z_2$  are linearly independent solutions to the differential equation, any linear combination of  $z_1$  and  $z_2$  is also a solution. Therefore  $y_1(t)$  and  $y_2(t)$  are solutions to the diff eq as well

$$y_1(t) = e^{4t} \cos(t)$$

$$y_2(t) = e^{4t} \sin(t)$$

- (c) (4 points) Determine the (real) solution with initial conditions  $y(0) = 4$  and  $y'(0) = -1$ .

$$y = C_1 e^{4t} \cos t + C_2 e^{4t} \sin t = e^{4t} (C_1 \cos t + C_2 \sin t)$$

$$y' = C_1 (4e^{4t} \cos t - e^{4t} \sin t) + C_2 (4e^{4t} \sin t + e^{4t} \cos t)$$

$$4 = C_1$$

$$-1 = 4C_1 + C_2$$

$$-1 = 16 + C_2$$

$$C_2 = -17$$

$$y = 4e^{4t} \cos t - 17e^{4t} \sin t$$

4. (13 points) Given that  $y_1(t) = t$  and  $y_2(t) = t^{-3}$  are solutions to the homogeneous equation

$$t^2 y'' + 3ty' - 3y = 0$$

Use variation of parameters to find the general solution to the inhomogeneous equation

$$t^2 y'' + 3ty' - 3y = \frac{1}{t}$$

of the form  $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ . (Remember: when calculating  $y_p'(t)$ , we set  $v_1' y_1 + v_2' y_2 = 0$ ).

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = \frac{1}{t^3}$$

$$y_p(t) = v_1 y_1 + v_2 y_2$$

$$y_p'(t) = \underbrace{v_1' y_1 + v_2' y_2}_0 + v_1 y_1' + v_2 y_2'$$

$$y_p''(t) = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''$$

$$y_1(t) = t \quad y_2(t) = t^{-3}$$

$$y_1'(t) = 1 \quad y_2'(t) = -3t^{-4}$$

$$W = \det \begin{pmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{pmatrix} = -3t^{-3} - t^{-3} = -4t^{-3}$$

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = \frac{1}{t^3}$$

$$v_2' = \frac{v_1 y_1'}{W} = \frac{t \left(\frac{1}{t^3}\right)}{-4t^{-3}} = \frac{t^{-2}}{-4t^{-3}} = \frac{t}{-4}$$

$$v_2 = \int -\frac{t}{4} dt = -\frac{t^2}{8}$$

$$v_1' = \frac{-y_2 y_1'}{W} = \frac{-t^{-3} \left(\frac{1}{t^3}\right)}{-4t^{-3}} = \frac{1}{4t^3}$$

$$v_1 = \int \frac{1}{4t^3} dt = -\frac{1}{8t^2}$$

$$y_p(t) = \left(-\frac{1}{8t^2}\right)(t) + \left(-\frac{t^2}{8}\right)(t^{-3}) = -\frac{1}{8t} - \frac{1}{8t} = \boxed{-\frac{1}{4t}}$$

$$y_p(t) = \frac{-1}{4t}$$

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