Midterm 1

Last Name: ______________________________

First Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Section:

	Tuesday:	Thursday:	
	2A	2B	TA: Yizhou Chen
	2C	2D	TA: Clark Lyons
	2E	2F	TA: Joel Barnett

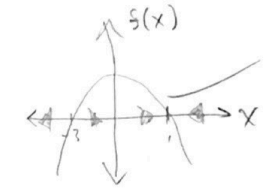
Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	13	13
2	13	12
3	12	12
4	12	12
Total:	50	49

13/13

1. Consider the differential equation



$$x' = -x^2 - 2x + 3$$

(a) (5 points) Determine the equilibrium solutions. Classify each as stable or unstable

5

$f(x) = -x^2 - 2x + 3 = 0$

$= x^2 + 2x - 3 = 0$

$= (x+3)(x-1) = 0$

$x = -3, \quad x = 1$

$x(t) = -3$ Unstable

$x(t) = 1$ Stable

(b) (2 points) If x is a particular solution with initial condition $x(t_0) > 1$, what is $\lim_{t\to\infty} x(t)$?

$\lim_{t\to\infty} x(t) = 1$

(c) (6 points) Determine the particular solution with initial condition $x(0) = 2$.

6

$\frac{dx}{dt} = -(x^2+2x-3) = -(x+3)(x-1)$

$\frac{dx}{(x+3)(x-1)} = -1 \cdot dt \qquad \frac{1}{(x+3)(x-1)} = \frac{A}{(x+3)} + \frac{B}{(x-1)} \Rightarrow 1 = A(x-1) + B(x+3)$

$A = -1/4 \quad B = 1/4$

$\int \frac{-1/4}{x+3}\,dx + \int \frac{1/4}{x-1}\,dx = -t + C$

$-\frac{1}{4}\ln(x+3) + \frac{1}{4}\ln(x-1) = -t + C$

$\frac{1}{4}\left(\ln(x-1) - \ln(x+3)\right) = -t + C$

$\frac{1}{4}\ln\left(\frac{x-1}{x+3}\right) = -t + C$

$\frac{1}{4}\ln\left(\frac{2-1}{2+3}\right) = C$

$\frac{1}{4}\ln\left(\frac{1}{5}\right) = C$

$\frac{1}{4}\ln\left(\frac{x-1}{x+3}\right) = -t + \frac{1}{4}\ln\left(\frac{1}{5}\right)$

$\ln\left(\frac{x-1}{x+3}\right) = -4t + \ln\left(\frac{1}{5}\right)$

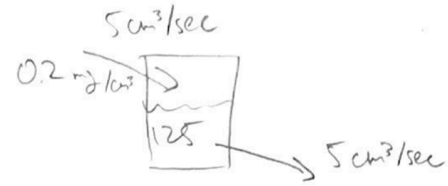
$\frac{x-1}{x+3} = e^{-4t} e^{\ln(\frac{1}{5})} = \frac{1}{5}e^{-4t}$

$x - 1 = \frac{1}{5}e^{-4t}(x+3) \Rightarrow x - 1 = x\frac{1}{5}e^{-4t} + \frac{3}{5}e^{-4t}$

$x - x\frac{1}{5}e^{-4t} = \frac{3}{5}e^{-4t} + 1$

$x\left(1 - \frac{1}{5}e^{-4t}\right) = \frac{3}{5}e^{-4t} + 1$

$x(t) = \frac{\frac{3}{5}e^{-4t} + 1}{1 - \frac{1}{5}e^{-4t}}$



2. Blood carries a drug to an organ at a rate of 5 cm^3/sec and leaves at the same rate. The organ has a liquid volume of 125 cm^3. The concentration of the drug in the blood entering the organ is 0.2 mg/cm^3. Let $x(t)$ denote the amount of the drug in the organ at time t.

   (a) (2 points) At what rate (in mg/sec) is the drug entering the organ?

   $$\frac{0.2\text{ mg}}{\text{cm}^3} \times \frac{5\text{ cm}^3}{\text{sec}} = 1\ \text{mg}/\text{sec}$$

   (b) (2 points) At what rate is the drug exiting the organ?

   $$\frac{5\text{ cm}^3}{\text{sec}} \times \frac{x(t)\text{ mg}}{125\text{ cm}^3} = \frac{1}{25}x(t)\ \text{mg}/\text{sec}$$

   25
   $5\overline{)125}$
   -10
   25

   (c) (5 points) Solve for $x(t)$ assuming that the person had no trace of the drug in their blood to start.

   $$x' = 1 - \frac{1}{25}x = -\frac{1}{25}x + 1 \qquad u = e^{-\int -\frac{1}{25}dt} = e^{\int \frac{1}{25}dt} = e^{\frac{1}{25}t}$$

   $$(xu)' = uf \Rightarrow xe^{\frac{1}{25}t} = \int e^{\frac{1}{25}t}\,dt = 25e^{\frac{1}{25}t} + C \Rightarrow xe^{\frac{1}{25}t} = 25e^{\frac{1}{25}t} + C$$

   $$x = 25 + \frac{C}{e^{\frac{1}{25}t}} \Rightarrow x(0) = 0 \Rightarrow 0 = 25 + \frac{C}{e^0} \Rightarrow 0 = 25 + C \Rightarrow C = -25$$

   $$x(t) = 25 - \frac{25}{e^{\frac{1}{25}t}}$$

3

   (d) (4 points) The person will begin feeling the effect of the drug when the concentration in the organ is 0.1 mg/cm^3. How long after taking the drug will the person feel its effect? (leave your answer exact, don't worry about a decimal approximation.)

   $$\frac{x(t)}{125} = 0.1$$

   $$0.1 = 25 - \frac{25}{e^{\frac{1}{25}t}} \Rightarrow -24.9 = \frac{-25}{e^{\frac{1}{25}t}} \Rightarrow (24.9)e^{\frac{1}{25}t} = 25$$

   $$e^{\frac{1}{25}t} = \frac{25}{24.9} \Rightarrow \frac{1}{25}t = \ln\left(\frac{25}{24.9}\right) \Rightarrow t = 25\ln\left(\frac{25}{24.9}\right)$$

3. Consider the differential equation

$$\frac{dx}{dt} = -(t + \cos t)x^2$$

(a) (5 points) Find the general solution.

$\frac{dx}{x^2} = -(t+\cos t)\,dt \Rightarrow \int x^{-2}\,dx = -\int t + \cos t\ dt \Rightarrow -x^{-1} = -\left[\frac{t^2}{2} + \sin t\right] + C$

$-\frac{1}{x} = -\frac{t^2}{2} - \sin t + C \Rightarrow \frac{1}{x} = \frac{t^2}{2} + \sin t + C \Rightarrow x(t) = \frac{1}{\frac{t^2}{2} + \sin t + C}$ ✓

(b) (2 points) Determine the particular solution with initial condition $x(0) = 1$.

$1 = \frac{1}{\frac{0}{2} + \sin 0 + C} = \frac{1}{0 + 0 + C} \Rightarrow C = 1$

$x(t) = \frac{1}{\frac{t^2}{2} + \sin t + 1}$ ✓

(c) (3 points) What is the interval of existence to the solution in part (b)? Explain.

$\frac{t^2}{2} + \sin t + 1 \neq 0$

$t^2 + 2\sin t + 2 \neq 0$

$\sin t$ ranges from $[-1, 1]$, so $2\sin t$ ranges from $[-2, 2]$. t^2 is always positive except when $t = 0$, but the expression still does not equal 0. There are no such values of t that the expression equals 0, thus

$(-\infty, \infty)$ ✓

(d) (2 points) Determine the particular solution with initial condition $x(0) = 0$

$0 = \frac{1}{\frac{0}{2} + \sin 0 + C} = \frac{1}{C}$

$x(t) = 0$ ✓

4. Consider the differential equation

$$y + (2y + kx)\frac{dy}{dx} = 0$$

(a) (4 points) What value of k makes the differential equation exact on the rectangle $(-\infty, \infty)$?

$$\underbrace{y}_{P}\,dx + \underbrace{(2y+kx)}_{Q}\,dy = 0$$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = k \qquad k = 1$$

(b) (5 points) Determine the general solution to the exact equation using the value of k found above. implicitly

$$y\,dx + (2y + x)\,dy = 0$$

$$\frac{\partial F}{\partial x} = P \Rightarrow F(x,y) = \int y\,dx = xy + \phi(y)$$

$$\frac{\partial F}{\partial y} = Q \Rightarrow x + \phi'(y) = 2y + x$$

$$\phi'(y) = 2y \qquad F(x,y) = xy + y^2 = C$$

$$\phi(y) = y^2$$

(c) (3 points) Determine the particular solution with initial condition $y(1) = -2$ (you may leave your answer implicitly defined).

$$xy + y^2 = C$$

$$(1)(-2) + (-2)^2 = C$$

$$-2 + 4 = C$$

12/12

$$C = 2$$

$$xy + y^2 = 2$$