

13 1. Consider the differential equation

$$x' = -x^2 - 2x + 3$$

(a) (5 points) Determine the equilibrium solutions. Classify each as stable or unstable

$$f(x) = -x^2 - 2x + 3$$

$$f'(x) = -2x - 2$$

$$x' = -(x^2 + 2x - 3)$$

$$x' = -(x-1)(x+3)$$

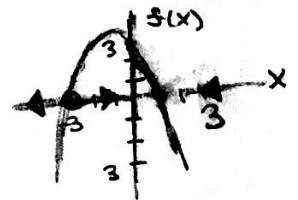
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$x=1$ $f'(1) < 0$ stable	$x=-3$ $f'(-3) > 0$ unstable
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(b) (2 points) If x is a particular solution with initial condition $x(t_0) > 1$, what is $\lim_{t \rightarrow \infty} x(t)$?

2 $\lim_{t \rightarrow \infty} x(t) = 1$

$x=1$ is a stable equilibrium point



(c) (6 points) Determine the particular solution with initial condition $x(0) = 2$.

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$$x' = -x^2 - 2x + 3$$

$$\frac{dx}{dt} = -(x-1)(x+3)$$

$$\frac{1}{(x-1)(x+3)} dx = -dt$$

$$\frac{1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{Ax+3A+Bx-3B}{(x-1)(x+3)} = \frac{(A+B)x + (3A-3B)}{(x-1)(x+3)}$$

$$A+B=0 \quad A=-B$$

$$3A-3B=1$$

$$4A=1 \quad A=\frac{1}{4} \quad B=-\frac{1}{4}$$

$$\frac{1}{4} \left(\frac{1}{x-1} - \frac{1}{x+3} \right) dx = \int -dt$$

$$\frac{1}{4} (\ln|x-1| - \ln|x+3|) = -t + C$$

$$\ln \frac{|x-1|}{|x+3|} = -4t + C$$

$$\frac{x-1}{x+3} = Ce^{-4t}$$

$$x-1 = (x+3)Ce^{-4t}$$

$$x(1 - Ce^{-4t}) = (3Ce^{-4t} + 1)$$

$$x = \frac{3Ce^{-4t} + 1}{1 - Ce^{-4t}}$$

$$x(0) = 2$$

$$2 = \frac{3C+1}{1-C}$$

$$2-2C = 3C+1$$

$$5C = 1$$

$$C = \frac{1}{5}$$

$$x(t) = \frac{3e^{-4t} + 1}{1 - e^{-4t}}$$

$$x(t) = \frac{3e^{-4t} + 1}{1 - e^{-4t}}$$

$$\lim_{t \rightarrow \infty} x(t) = \frac{1}{1-0} = 1$$

2. Blood carries a drug to an organ at a rate of $5 \text{ cm}^3/\text{sec}$ and leaves at the same rate. The organ has a liquid volume of 125 cm^3 . The concentration of the drug in the blood entering the organ is $0.2 \text{ mg}/\text{cm}^3$. Let $x(t)$ denote the amount of the drug in the organ at time t .

(a) (2 points) At what rate (in mg/sec) is the drug entering the organ?

$$\text{rate in} = \text{volume in} \times \text{concentration in} = 5 \frac{\text{cm}^3}{\text{s}} \times 0.2 \frac{\text{mg}}{\text{cm}^3} = 1 \text{ mg/s}$$

(b) (2 points) At what rate is the drug exiting the organ?

$$\text{rate out} = \text{volume out} \times \text{concentration out} = \frac{5 \text{ cm}^3}{\text{s}} \times \frac{x(t) \text{ mg}}{125 \text{ cm}^3} = \frac{x}{25} \text{ mg/s}$$

(c) (5 points) Solve for $x(t)$ assuming that the person had no trace of the drug in their blood to start.

$$\text{rate in} - \text{rate out} = 1 - \frac{x}{25}$$

$$x' = 1 - \frac{x}{25}$$

$$u = e^{-\int \frac{1}{25} dt} = e^{-\frac{t}{25}} = e^{-\frac{t}{25}}$$

$$(ux)' = 1(u)$$

$$ux = \int e^{-\frac{t}{25}} dt$$

$$ux = 25e^{-\frac{t}{25}} + C$$

$$x = \frac{1}{u} (25e^{-\frac{t}{25}} + C)$$

$$x = 25 + Ce^{-\frac{t}{25}}$$

$$x(0) = 0$$

$$0 = 25 + C$$

$$C = -25$$

$$x(t) = 25 - 25e^{-\frac{t}{25}}$$

$$x(t) = 25(1 - e^{-\frac{t}{25}})$$

(d) (4 points) The person will begin feeling the effect of the drug when the concentration in the organ is $0.1 \text{ mg}/\text{cm}^3$. How long after taking the drug will the person feel its effect? (leave your answer exact, don't worry about a decimal approximation.)

let $t_f =$ time when drug is felt
concentration = $\frac{x}{V}$

$$c = \frac{x}{V}$$

$$x = Vc = (125 \text{ cm}^3)(0.1 \text{ mg}/\text{cm}^3) = 12.5 \text{ mg}$$

$$c(t_f) = 0.1 \text{ mg}/\text{cm}^3$$

$$x(t_f) = 0.1 \frac{\text{mg}}{\text{cm}^3} \cdot 125 \text{ cm}^3 = 12.5 \text{ mg}$$

$$12.5 = 25(1 - e^{-\frac{t_f}{25}})$$

$$\frac{1}{2} = 1 - e^{-\frac{t_f}{25}}$$

$$e^{-\frac{t_f}{25}} = \frac{1}{2}$$

$$-\frac{t_f}{25} = \ln\left(\frac{1}{2}\right)$$

$$t_f = 25 \ln(2) \text{ seconds}$$

3. Consider the differential equation

$$\frac{dx}{dt} = -(t + \cos t)x^2$$

(a) (5 points) Find the general solution.

$$\frac{dx}{x^2} = -(t + \cos t)x^2$$

$$\int \frac{1}{x^2} dx = \int (t + \cos t) dt$$

$$\frac{1}{x} = \frac{t^2}{2} + \sin t + C$$

$$x(t) = \frac{1}{\frac{t^2}{2} + \sin t + C}$$

(b) (2 points) Determine the particular solution with initial condition $x(0) = 1$.

$$x(0) = 1$$

$$1 = \frac{1}{0 + C}$$

$$1 = \frac{1}{C} \quad C = 1$$

$$x(t) = \frac{1}{\frac{t^2}{2} + \sin t + 1}$$

(c) (3 points) What is the interval of existence to the solution in part (b)? Explain.

$$\frac{t^2}{2} + \sin t + 1 \neq 0$$

always ≥ 1

$$\frac{t^2}{2} + \sin t + 1 = 0 \quad -\sin t = \left(\frac{t^2}{2} + 1\right)$$

$$\frac{t^2}{2} + \sin t = -1$$

$\sin t$ is always between -1 and 1

$$\frac{t^2}{2} \text{ is always } > 0$$

$\sin(t)$ would have to be less than -1 for some point

for $\frac{t^2}{2} + \sin(t) + 1$ to be 0 so $\frac{t^2}{2} + \sin(t) + 1$ is never 0

Interval of Existence
 $(-\infty, \infty)$

$-1 \leq \sin t \leq 1$ & $\frac{1}{2}t^2 \geq 0$
So the sine function is the only possible way for our denominator to vanish

Only can vanish if $\sin(t) = -1$
 $t = -\frac{3\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{7\pi}{2}$ or $\frac{11\pi}{2}$

(d) (2 points) Determine the particular solution with initial condition $x(0) = 0$

~~$$x(0) = 0$$~~

~~$$0 = \frac{1}{0 + C}$$~~

no solution

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4. Consider the differential equation

$$y + (2y + kx) \frac{dy}{dx} = 0$$

- (a) (4 points) What value of k makes the differential equation exact on the rectangle $(-\infty, \infty)$?

$$y dx + (2y + kx) dy = 0$$

$$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = k$$

IF EXACT $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, SO

IF $k=1$, the diff eq is exact

$$\boxed{k=1}$$

- (b) (5 points) Determine the general solution to the exact equation using the value of k found above.

$$F(x,y) = \int P(x,y) dx + \phi(y) = \int y dx + \phi(y) = xy + \phi(y) \quad 12/12$$

$$Q(x,y) = \frac{\partial F}{\partial y} = x + \phi'(y)$$

$$Q(x,y) = 2y + x$$

$$2y + x = x + \phi'(y)$$

$$\phi'(y) = 2y$$

$$\phi(y) = y^2$$

$$\boxed{F(x,y) = xy + y^2 = C}$$

- (c) (3 points) Determine the particular solution with initial condition $y(1) = -2$ (you may leave your answer implicitly defined).

$$y(1) = -2$$

$$(1)(-2) + (-2)^2 = C$$

$$-2 + 4 = C$$

$$C = 2$$

$$\boxed{xy + y^2 = 2}$$