# Final Exam

Please circle or box your final answers. You may not use any other resources, including collaborating with other students. **The use of any such additional resources is a form of academic dishonesty and may result in your exam being void.** When uploading to Gradescope, please submit each question on a separate page.

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Please do not write below this line.

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 10 | |
| 4 | 20 | |
| 5 | 16 | |
| 6 | 10 | |
| 7 | 14 | |
| 8 | 20 | |
| Total: | 100 | |

1. (5 points) Suppose the first order linear differential equation $x' + p(t)x = f(t)$ has an integrating factor $v(t) = \sec(t)$. What is $p(t)$?

$$v(t) = \sec(t) = e^{\int p(t)\,dt}$$

$$\ln(\sec(t)) = \int p(t)\,dt$$

$$\frac{d}{dt}\left(\ln(\sec(t))\right) = \frac{d}{dt}\left(\int p(t)\,dt\right)$$

$$\boxed{p(t) = \frac{1}{\sec(t)} \cdot \frac{-\sin(t)}{-\cos^2(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t)}$$

2. (5 points) A recipe calls for a gallon of salt water solution containing 5 ounces of a salt. In your 5 gallon container, you accidentally create a solution with twice that concentration. Suppose you can pour pure water into your container at a rate of 1 gallon per minute while draining it at the same rate. How long will you have to do this to bring your container to the correct concentration? (You may use a calculator to round your answer to one decimal place).

$x(t)$ = total oz salt in 5 gallon container

$$x'(t) = \frac{-x(t)}{5} \qquad x(t) = Ae^{\int -\frac{1}{5}dt} = Ae^{-\frac{1}{5}t}$$

initial: $\frac{5\,oz}{1\,gal} \cdot 5\,gal \cdot 2 = 50\,oz$ $\quad x(0) = A = 50$

goal: $\frac{5\,oz}{1\,gal} \cdot 5\,gal = 25\,oz$

$$x(t) = 50e^{-\frac{1}{5}t}$$

$$25 = 50e^{-\frac{1}{5}t}$$

$$\frac{1}{2} = e^{-\frac{1}{5}t}$$

$$\ln\left(\tfrac{1}{2}\right) = -\frac{1}{5}t$$

$$\boxed{t = 5\ln(2) \approx 3.5 \text{ minutes}}$$

3. (a) (5 points) Consider the differential equation

$$y'' + 2y' - y = -5\sin(2t)$$

Let $y_p(t) = A\cos(2t) + B\sin(2t)$ and use the method of undetermined coefficients to calculate $A$ and $B$. State the particular solution.

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t) \qquad y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

$y_p'' + 2y_p' - y_p = -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) - A\cos(2t) - B\sin(2t)$

$= (-4A + 4B - A)\cos(2t) + (-4B - 4A - B)\sin 2t$

$= (-5A + 4B)\cos(2t) + (-4A - 5B)\sin 2t = -5\sin(2t)$

$\begin{bmatrix} -5 & 4 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -4 & -5 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -5 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} -5 & -4 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{20}{41} \\ \frac{25}{41} \end{bmatrix}$

$\boxed{A = \frac{20}{41} \quad B = \frac{25}{41} \quad y_p(t) = \frac{20}{41}\cos(2t) + \frac{25}{41}\sin(2t)}$

(b) (5 points) Consider the differential equation

$$z'' + 2z' - z = -5e^{2it}$$

Let $z_p = ae^{2it}$ and use the method of undetermined coefficients to calculate $a$ (simplify so that $a$ has real denominator, if necessary). State the (complex) particular solution in the form $x(t) + iy(t)$.

$z_p' = 2aie^{2it} \qquad z_p'' = -4ae^{2it}$

$z_p'' + 2z_p' - z_p = -4ae^{2it} + 4aie^{2it} - ae^{2it} = -5ae^{2it} + 4aie^{2it} = -5e^{2it}$

$a(-5 + 4i) = -5$

$a = \frac{-5}{-5 + 4i} = \frac{-5(-5 - 4i)}{(-5 + 4i)(-5 - 4i)} = \boxed{\frac{25 + 20i}{41}}$

$z_p = \left(\frac{25 + 20i}{41}\right)e^{2it} = \left(\frac{25 + 20i}{41}\right)(\cos(2t) + i\sin(2t))$

$= \frac{25}{41}\cos(2t) + \frac{25}{41}i\sin(2t) + \frac{20}{41}i\cos(2t) - \frac{20}{41}\sin(2t)$

$\boxed{= \left(\frac{25}{41}\cos(2t) - \frac{20}{41}\sin(2t)\right) + i\left(\frac{20}{41}\cos(2t) + \frac{25}{41}\sin(2t)\right)}$

4. (20 points) For each the systems below:

$$x' = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} x, \qquad y' = \begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix} y$$

(a) (4 points) Give the characteristic polynomial and calculate the eigenvalues of the matrix.

(b) (4 points) Draw the phase plane portrait . Label the direction of motion (forward time), all eigenvectors, and the direction of rotation, if applicable.

(c) (2 point) Classify the equilibrium solution.

a. $x' = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} x$

$T = 4 \quad D = 3$

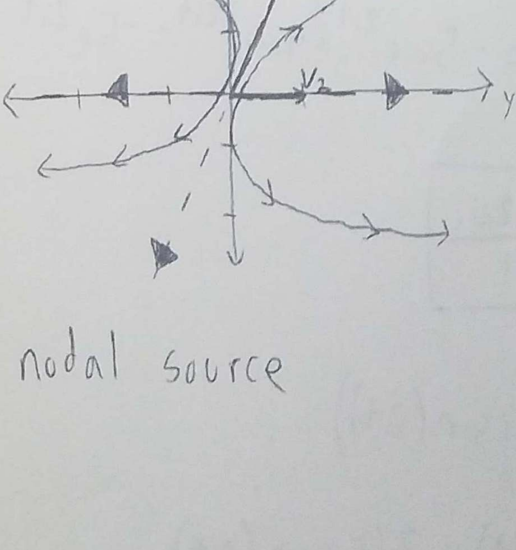
$p(s) = s^2 - 4s + 3 = 0$

$s = 1, 3$

$s_1 = 1 \quad \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$s_2 = 3 \quad \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b.




c.

nodal source

a. $y' = \begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix} y$

$T = 8 \quad D = 17$

$p(s) = s^2 - 8s + 17 = 0$

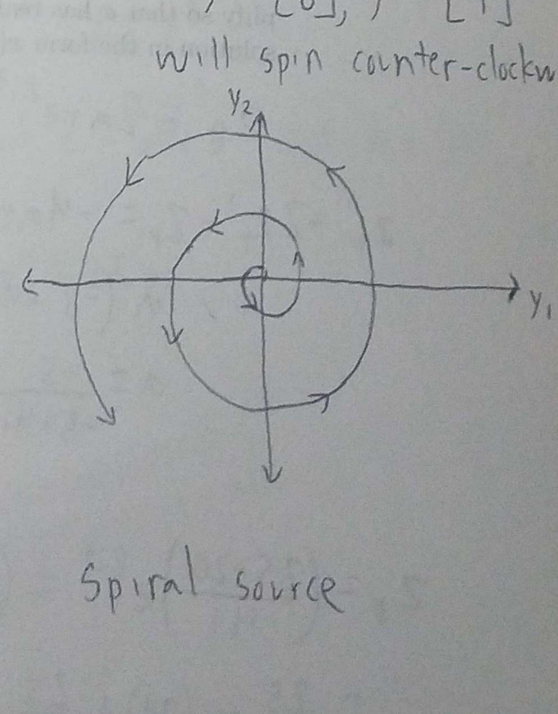
$s = \frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm \sqrt{-4}}{2}$

$= 4 \pm i$

$\alpha = 4 > 0$

b, at $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y' = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

will spin counter-clockwise




c.

Spiral source

5. Let $c$ be a real number and consider the system

$$y' = \begin{pmatrix} c & 0 \\ 1 & 1 \end{pmatrix} y$$

(a) (4 points) For what values of $c$ will the equilibrium solution be a saddle point?

(b) (4 points) For what values of $c$ will the equilibrium solution be a nodal source?

(c) (4 points) What values of $c$ give *nongeneric* equilibrium solutions?

(d) (4 points) Explain why no value of $c$ will give a nodal or spiral sinks.

$T = c+1 \qquad D = c$

a. saddle points: $D < 0$, so $\boxed{c < 0}$

b. nodal source: $T > 0$, $0 < D < \frac{1}{4}T^2$

so $c+1 > 0$, $c > 0$, $c < \frac{1}{4}((c+1))^2$

$c < \frac{1}{4}c^2 + \frac{1}{2}c + \frac{1}{4}$

$0 < \frac{1}{4}c^2 - \frac{1}{2}c + \frac{1}{4}$

$0 < c^2 - 2c + 1 = (c-1)^2$

$c \neq 1$

$\boxed{\text{will be nodal source for all } c > 0 \text{ where } c \neq 1}$

c. nongeneric equilibrium solutions: $D = 0$, $T = 0$ and $D \geq 0$, $D = \frac{1}{4}T^2$

$c = 0$     ↑ impossible     $c = 1$

$\boxed{c = 0 \text{ and } c = 1 \text{ give nongeneric equilibrium solutions}}$

d. For nodal or spiral sinks to be even possible, $T < 0$ and $D > 0$, so $T < D$, but $T = c+1$ and $D = c$, so $T = D+1$, so it is impossible for $T < D$ so nodal or spiral sinks are impossible

6. Consider the system

$$y' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y$$

(a) (3 points) Show that the system has infinitely many equilibrium (i.e. constant) solutions.

(b) (3 points) What is the dimension of the nullspace of $A$?

(c) (4 points) Find a fundamental set of solutions (write your final answers explicitly, without exponential matrices).

a. $y' = 0 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y \quad y = \begin{bmatrix} c \\ c \end{bmatrix}$ where $c$ is any constant

so any multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an equilibrium solution, thus there are infinite equilibrium solutions

b. $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad v = \begin{bmatrix} c \\ c \end{bmatrix}$

nullspace of A is span$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, so has dimension 1

c. $p(\lambda) = \lambda^2 = 0 \quad \lambda = 0$

$\lambda_1 = 0 \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y_1(t) = e^{0t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y_2(t) = e^{0t}(v_2 + t v_1) = v_2 + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A - \lambda I)v_2 = v_1 \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad v_2 = \begin{bmatrix} c+1 \\ c \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y_2(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+t \\ t \end{bmatrix}$

$$\boxed{y(t) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1+t \\ t \end{bmatrix}}$$

7. Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) (4 points) Give the characteristic polynomial of $A$ and calculate the eigenvalues of $A$.

(b) (4 points) Determine the dimension of the nullspace of $(A - \lambda_i I)$ for each eigenvalue $\lambda_i$ found above.

(c) (6 points) Find a fundamental set of solutions to the system $y' = Ay$ (write your final answers explicitly, without exponential matrices)

a. $p(\lambda) = -\lambda\left((1-\lambda)(-1-\lambda)+1\right) = -\lambda\left(-1-\lambda+\lambda+\lambda^2+1\right) = -\lambda^3 = 0$

$\boxed{\lambda = 0}$

b. $\lambda = 0$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $v = \begin{bmatrix} -c \\ c \\ 0 \end{bmatrix}$ nullspace of A is Span$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$, so it is of dimension 1

c. $\lambda_1 = 0$ $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $y_1 = e^{0t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$v = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are in null space of $A^2$, but both of them together are not linearly independent with $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, can only use one, use $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $y_2 = e^{At}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ all vectors are in nullspace of $A^3$, choose $v_3$ linearly independent from $v_1$ and $v_2$

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $y_3 = e^{At}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\boxed{y(t) = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} t+1 \\ -t \\ 0 \end{bmatrix} + c_3\begin{bmatrix} t^2+2t \\ -t^2 \\ 2 \end{bmatrix}}$

8. (20 points) True or False (2 points each). All matrices are assumed to have real entries.

(a) (**True** or **False**) The differential equation $P(x,y)dx - Q(x,y)dy = 0$ is exact if

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

(b) (**True** or **False**) If the eigenvalues of a $2 \times 2$ matrix $A$ are both positive, then the nonequilibrium solutions to $x' = Ax$ all satisfy $\lim_{t\to\infty} x(t) = 0$.

(c) (**True** or **False**) If $x_0$ is an equilibrium solution for an autonomous equation $x' = f(x)$, then $x_0$ is stable if $f'(x_0) > 0$.

(d) (**True** or **False**) A $3 \times 3$ matrix cannot have all complex eigenvalues.

(e) (**True** or **False**) The function $v(t) = e^{-\int p(t)dt}$ is an integrating factor for

$$\frac{dx}{dt} + p(t)x = f(t)$$

(f) (**True** or **False**) There is a polynomial solution to $y' = 4 - y^2$.

(g) (**True** or **False**) If $A$ is any $n \times n$ matrix and $v$ is any $n$-vector, then $y(t) = e^{tA}v$ is a solution to the linear system $y' = Ay$.

(h) (**True** or **False**) If $A, B$ are $2 \times 2$ matrices with $\det(A) = -\det(B)$ and $\text{trace}(A) = -\text{trace}(B)$, then either $x' = Ax$ or $x' = Bx$ has a saddle point equilibrium solution.

(i) (**True** or **False**) If $A$ is a $2 \times 2$ matrix with $\det(A) = 0$, then $x' = Ax$ has infinitely many equilibrium solutions

(j) (**True** or **False**) The level set $xy + y^2 = c$ solves the differential equation

$$ydx + (2y + x)dy = 0$$