20W-MATH33B-2 Final Exam

ANDREW NG

TOTAL POINTS

100 / 100

QUESTION 1

1 Integrating Factor Question 5 / 5

✓ **- 0 pts** Correct

- **1 pts** Dropped/introduced minus sign
- **5 pts** No progress
- **4 pts** Major errors

QUESTION 2

2 Mixing Question 5 / 5

✓ **- 0 pts** Correct

- **1 pts** Minor error
- **2 pts** Correct S(t) but wrong final answer
- **3 pts** Initial value error
- **4 pts** Major errors
- **5 pts** No progress

QUESTION 3

Undetermined Coefficients 10 pts

3.1 Real Method 5 / 5

✓ **- 0 pts** Correct

- **1 pts** minor error
- **2 pts** Click here to replace this description.
- **3 pts** -3
- **4 pts** Click here to replace this description.
- **5 pts** -5

3.2 Complex Method 5 / 5

✓ **- 0 pts** Correct

- **1 pts** Click here to replace this description.
- **2 pts** Click here to replace this description.
- **3 pts** Click here to replace this description.
- **4 pts** Click here to replace this description.
- **5 pts** Click here to replace this description.

QUESTION 4

2x2 Systems 20 pts

4.1 Eigenvalues 1 4 / 4

✓ **- 0 pts** Correct

- **2 pts** characteristic polynomial incorrect
- **2 pts** eigenvalue incorrect
- **1 pts** one eigenvalue incorrect

4.2 Phase Plane 1 4 / 4

✓ **- 0 pts** Correct

- **2 pts** both eigenvectors incorrect
- **1 pts** one wrong eigenvector
- **1 pts** direction of motion or trajectory incorrect
- **1 pts** eigenvector on the plane incorrect

4.3 Equilibrium 1 2 / 2

✓ **- 0 pts** Correct

- **2 pts** incorrect type
- **1 pts** incomplete

4.4 Eigenvalues 2 4 / 4

✓ **- 0 pts** Correct

- **2 pts** characteristic polynomial incorrect
- **1 pts** eigenvalues error

4.5 Phase Plane 2 4 / 4

✓ **- 0 pts** Correct

- **2 pts** eigenvectors incorrect
- **1 pts** direction of motion incorrect

4.6 Equilibrium 2 2 / 2

✓ **- 0 pts** Correct

- **1 pts** not specific enough
- **1 pts** error

QUESTION 5

Unknown c 16 pts

5.1 Saddle Point 4 / 4

✓ **- 0 pts** Correct

- 2 pts Error

- 4 pts Incorrect

5.2 Nodal Source 4 / 4

✓ **- 0 pts** Correct

- 2 pts Error

- 4 pts Incorrect

- 1 pts Minor Error

5.3 Nongeneric 4 / 4

✓ **- 0 pts** Correct

- 2 pts Error

- 1 pts Whoops

- 4 pts Incorrect

5.4 No Sinks 4 / 4

✓ **- 0 pts** Correct

- 1 pts Error

- 2 pts Not Complete

- 4 pts Click here to replace this description.

QUESTION 6

2x2 Exponential 10 pts

6.1 Infinitely Many Solutions 3 / 3

✓ **- 0 pts** Correct

- 2 pts You need to explain why there are infinitely many equilibrium solutions besides just stating that the matrix is singular.

- 3 pts Incorrect

6.2 Dimension of Nullspace 3 / 3

✓ **- 0 pts** Correct

- 3 pts Incorrect

6.3 Fundamental Set 4 / 4

✓ **- 0 pts** Correct

- 1 pts Minor error

- 2 pts Found eigenvector and generalized eigenvector, but wrong final answer

- 3 pts Major errors

- 4 pts No progress

QUESTION 7

3x3 Exponential 14 pts

7.1 Characteristic Polynomial & Eigenvalues 4 / 4

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

- 4 pts Click here to replace this description.

7.2 Dimension of Nullspaces 4 / 4

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

- 4 pts Click here to replace this description.

7.3 Fundamental Set 6 / 6

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

- 4 pts Click here to replace this description.

- 5 pts Click here to replace this description.

- 6 pts Click here to replace this description.

QUESTION 8

8 True/False 20 / 20

- 2 pts a incorrect

- 2 pts b incorrect

- 2 pts c incorrect

- 2 pts d incorrect

- 2 pts e incorrect

- 2 pts f incorrect

- 2 pts g incorrect

- 2 pts h incorrect

- **- 2 pts** i incorrect
- **- 2 pts** j incorrect
- ✓ **- 0 pts** all correct
- **- 20 pts** can't find #8 page

1. (5 points) Suppose the first order linear differential equation $x' + p(t)x = f(t)$ has an integrating factor $v(t) = \sec(t)$. What is $p(t)$?

$x'(t) + (-p(t))x = f(t)$

$v(t) = e^{-\int -p(t)dt} = e^{\int p(t)dt} = \sec(t)$

$\int p(t)dt = \ln(\sec(t))$

$p(t) = \frac{d}{dt}\ln(\sec(t)) = \frac{1}{\sec(t)}(\sec(t)\tan(t)) = \boxed{\tan(t)}$

$\cos t \neq 0$ $\cos t \neq 0$

2. (5 points) A recipe calls for a gallon of salt water solution containing 5 ounces of a salt. In your 5 gallon container, you accidentally create a solution with twice that concentration. Suppose you can pour pure water into your container at a rate of 1 gallon per minute while draining it at the same rate. How long will you have to do this to bring your container to the correct concentration? (You may use a calculator to round your answer to one decimal place).

target: $\frac{5 \text{ ounces}}{\text{gallon}}$, since $V = 5$, target = 25 ounces

let s = salt content (ounces)
t = time (min)
V = volume (gal)
C = concentration (ounces/gal)

$\frac{ds}{dt} = -\underbrace{\frac{1 \text{ gal}}{\text{min}} \times \frac{s \text{ ounces}}{V \text{ gal}}}_{\text{out}} + \underbrace{\frac{1 \text{ gal}}{\text{min}} \times \frac{0 \text{ ounces}}{V \text{ gal}}}_{\text{in}}$

$\frac{ds}{dt} = -\frac{s}{V} \text{ ounces/gallon} = -\frac{s}{5} \frac{\text{ounces}}{\text{gallon}}$

$\int \frac{ds}{s} = \int -\frac{1}{5} dt$

$\ln|s| = -\frac{1}{5}t + K$

$|s| = e^{-\frac{1}{5}t + K} = e^K e^{-t/5} = Ae^{-t/5}$

$s = Be^{-t/5}$

$s_0 = \frac{10 \text{ ounces}}{\text{gal}} \times 5\text{gal} = 50 \text{ ounces}$

$s(0) = Be^0 = B \Rightarrow B = 50$

$s(t) = 50e^{-t/5}$

to find time: $25 = 50e^{-t/5} \Rightarrow \frac{1}{2} = e^{-t/5}$

$\ln(\frac{1}{2}) = -t/5 \Rightarrow t = -5\ln(\frac{1}{2}) \approx$ 3.5 minutes

1. (5 points) Suppose the first order linear differential equation $x' + p(t)x = f(t)$ has an integrating factor $v(t) = \sec(t)$. What is $p(t)$?

$x'(t) + (-p(t))x = f(t)$

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$p(t) = \frac{d}{dt}\ln(\sec(t)) = \frac{1}{\sec(t)}(\sec(t)\tan(t)) = \boxed{\tan(t)}$

$\cos t \neq 0$ $\quad$ $\cos t \neq 0$

2. (5 points) A recipe calls for a gallon of salt water solution containing 5 ounces of a salt. In your 5 gallon container, you accidentally create a solution with twice that concentration. Suppose you can pour pure water into your container at a rate of 1 gallon per minute while draining it at the same rate. How long will you have to do this to bring your container to the correct concentration? (You may use a calculator to round your answer to one decimal place).

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$\ln(\frac{1}{2}) = -t/5 \Rightarrow t = -5\ln(\frac{1}{2}) \approx$ 3.5 minutes

3. (a) (5 points) Consider the differential equation

$$y'' + 2y' - y = -5\sin(2t)$$

Let $y_p(t) = A\cos(2t) + B\sin(2t)$ and use the method of undetermined coefficients to calculate A and B. State the particular solution.

First, check general: $y''+2y'-y=0 \Rightarrow \lambda^2+2\lambda-1 \Rightarrow \lambda = \frac{-2\pm\sqrt{4+4}}{2} = -1\pm 2\sqrt{2} \rightarrow e^{(-1+2\sqrt{2})t}, e^{(-1-2\sqrt{2})t} \Rightarrow$ particular not part of homogeneous

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

$y''+2y'-y = -4A\cos(2t) - 4B\sin(2t) + 2(-2A\sin(2t) + 2B\cos(2t)) - (A\cos(2t) + B\sin(2t))$

$= -4A\cos(2t) - 4B\sin(2t) - 4A\sin 2t + 4B\cos(2t) - A\cos(2t) - B\sin(2t)$

$= (-4A + 4B - A)\cos(2t) + (-4B - 4A - B)\sin(2t) = -5\sin 2t$

$-5A + 4B = 0$ $\qquad$ $-5B - 4A = -5$

$-20A + 16B = 0$

$-20A - 25B = -25$

$41B = 25 \Rightarrow B = \frac{25}{41} \Rightarrow -5A + 4(\frac{25}{41}) = 0 \Rightarrow A = \frac{4}{5}(\frac{25}{41}) = \frac{20}{41}$

$y_p(t) = \frac{20}{41}\cos(2t) + \frac{25}{41}\sin(2t)$

(b) (5 points) Consider the differential equation

$$z'' + 2z' - z = -5e^{2it}$$

Let $z_p = ae^{2it}$ and use the method of undetermined coefficients to calculate a (simplify so that a has real denominator, if necessary). State the (complex) particular solution in the form $x(t) + iy(t)$.

$z_p' = 2iae^{2it}$

$z_p'' = -4ae^{2it}$

$z_p'' + 2z_p' - z_p = -4ae^{2it} + 4iae^{2it} - ae^{2it} = -5ae^{2it} + 4iae^{2it} = -5e^{2it}$

$-5a + 4ia = -5 \Rightarrow (-5+4i)a = -5$

$a = \frac{-5}{-5+4i} \times \frac{-5-4i}{-5-4i} = \frac{-5(-5-4i)}{25+16}$

$a = \frac{25+20i}{41}$

$z_p = \left(\frac{25+20i}{41}\right)e^{2it} = \left(\frac{25}{41} + \frac{20i}{41}\right)(\cos(2t) + i\sin(2t))$

$= \frac{25}{41}\cos(2t) + \frac{25}{41}i\sin(2t) + \frac{20i}{41}\cos(2t) - \frac{20}{41}\sin(2t)$

$z_p = \left(\frac{25}{41}\cos(2t) - \frac{20}{41}\sin(2t)\right) + i\left(\frac{25}{41}\sin(2t) + \frac{20}{41}\cos(2t)\right)$

3. (a) (5 points) Consider the differential equation

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$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

$y'' + 2y' - y = -4A\cos(2t) - 4B\sin(2t) + 2(-2A\sin(2t) + 2B\cos(2t)) - (A\cos(2t) + B\sin(2t))$

$= -4A\cos(2t) - 4B\sin(2t) - 4A\sin 2t + 4B\cos(2t) - A\cos(2t) - B\sin(2t)$

$= (-4A + 4B - A)\cos(2t) + (-4B - 4A - B)\sin(2t) = -5\sin 2t$

$-5A + 4B = 0 \qquad -5B - 4A = -5$

$-20A + 16B = 0$

$-20A - 25B = -25$

$41B = 25 \Rightarrow B = \frac{25}{41} \Rightarrow -5A + 4\left(\frac{25}{41}\right) = 0 \Rightarrow A = \frac{4}{5}\left(\frac{25}{41}\right) = \frac{20}{41}$

$$\boxed{y_p(t) = \frac{20}{41}\cos(2t) + \frac{25}{41}\sin(2t)}$$

(b) (5 points) Consider the differential equation

$$z'' + 2z' - z = -5e^{2it}$$

Let $z_p = ae^{2it}$ and use the method of undetermined coefficients to calculate a (simplify so that a has real denominator, if necessary). State the (complex) particular solution in the form $x(t) + iy(t)$.

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$z_p'' = -4ae^{2it}$

$z_p'' + 2z_p' - z_p = -4ae^{2it} + 4iae^{2it} - ae^{2it} = -5ae^{2it} + 4iae^{2it} = -5e^{2it}$

$-5a + 4ia = -5 \Rightarrow (-5 + 4i)a = -5$

$a = \frac{-5}{-5+4i} \times \frac{-5-4i}{-5-4i} = \frac{-5(-5-4i)}{25+16}$

$$\boxed{a = \frac{25+20i}{41}}$$

$z_p = \left(\frac{25+20i}{41}\right)e^{2it} = \left(\frac{25}{41} + \frac{20i}{41}\right)(\cos(2t) + i\sin(2t))$

$= \frac{25}{41}\cos(2t) + \frac{25}{41}i\sin(2t) + \frac{20i}{41}\cos(2t) - \frac{20}{41}\sin(2t)$

$$\boxed{z_p = \left(\frac{25}{41}\cos(2t) - \frac{20}{41}\sin(2t)\right) + i\left(\frac{25}{41}\sin(2t) + \frac{20}{41}\cos(2t)\right)}$$

4. (20 points) For each the systems below:

$$x' = \underbrace{\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}}_{A} x, \qquad y' = \underbrace{\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix}}_{B} y$$

(a) (4 points) Give the characteristic polynomial and calculate the eigenvalues of the matrix.

no eigenvectors ?

(b) (4 points) Draw the phase plane portrait . Label the direction of motion (forward time), all eigenvectors, and the direction of rotation, if applicable.

(c) (2 point) Classify the equilibrium solution.

For the first one:

(a) $T(A) = 4 \quad det(A) = 3$

$p(\lambda) = \lambda^2 - T(A)\lambda + det(A)$

$\boxed{p(\lambda) = \lambda^2 - 4\lambda + 3}$

$= (\lambda - 1)(\lambda - 3) \Rightarrow \boxed{\lambda_1 = 1,\ \lambda_2 = 3}$

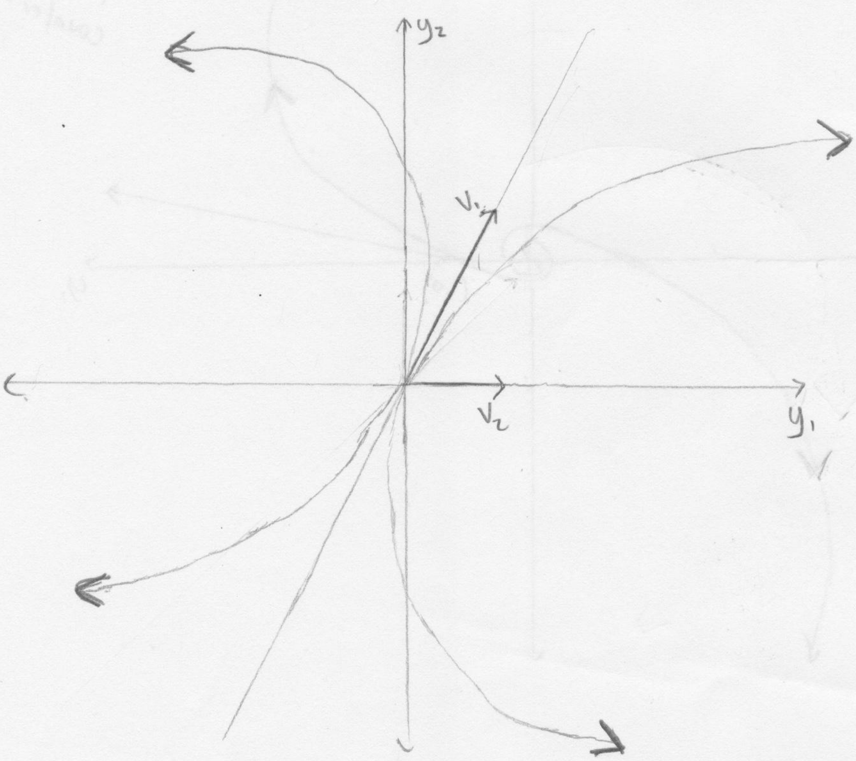
finding eigenvectors:

$\lambda_1 = 1 \Rightarrow A - I = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = 3 \Rightarrow A - 3I = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b)



as $t \to \infty$, $y \to$ direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$t \to -\infty$ $y \to$ direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $\boxed{\text{nodal source}}$ because $T^2 - 4D = 16 - 12 = 4 > 0$

and $D = 3 > 0$ and $T = 4 > 0$

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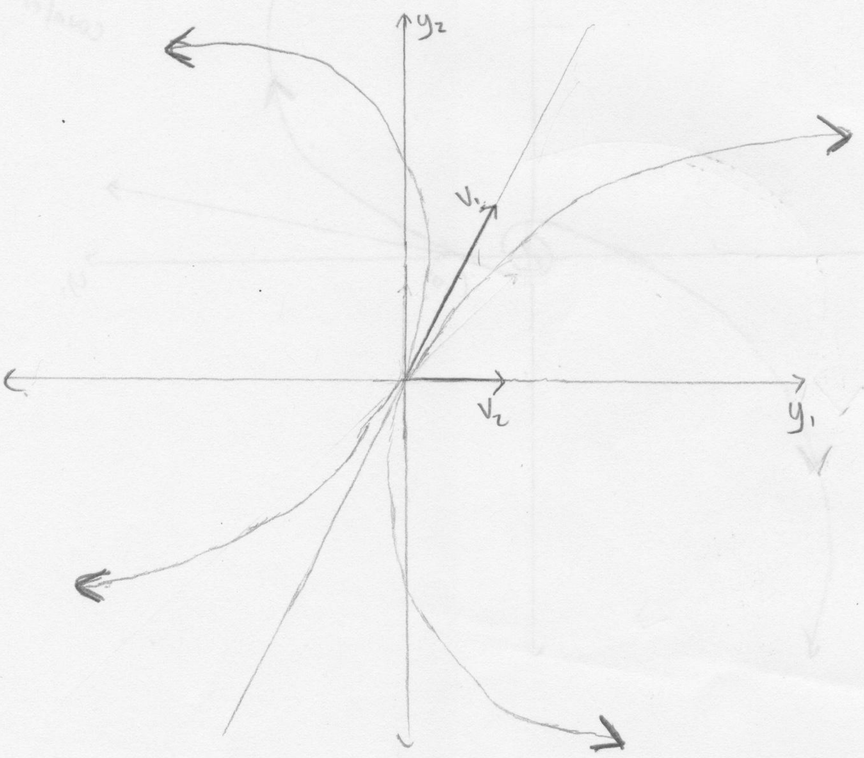
$\lambda_2 = 3 \Rightarrow A - 3I = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b)



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$= (\lambda-1)(\lambda-3) \Rightarrow \boxed{\lambda_1 = 1, \ \lambda_2 = 3}$

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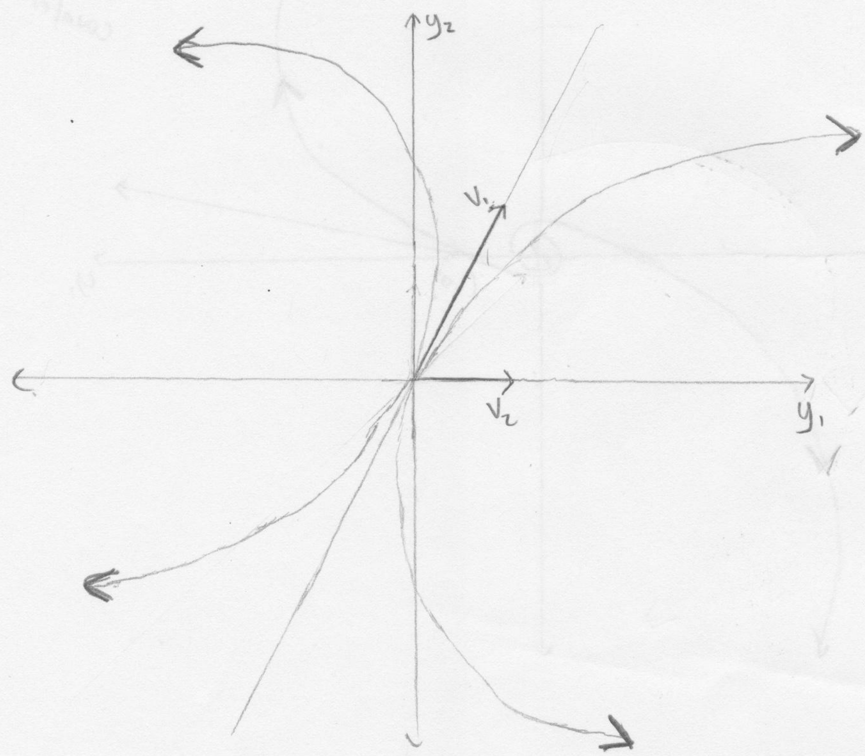
$y(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b)



as $t \to \infty$, $y \to$ direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$t \to -\infty$ $y \to$ direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $\boxed{\text{nodal source}}$ because $T^2 - 4D = 16 - 12 = 4 > 0$

and $D = 3 > 0$ and $T = 4 > 0$

(q4) for $y' = \underbrace{\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix}}_{B} y$

(a) $T(B) = 8$

$D(B) = 12 + 5 = 17$

$\boxed{p(\lambda) = \lambda^2 - 8\lambda + 17}$

$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm \frac{\sqrt{-4}}{2} = 4 \pm i$

$\boxed{\begin{array}{l} \lambda_1 = 4 - i \\ \lambda_2 = 4 + i \end{array}}$

(b) $\lambda_1 = 4 - i \Rightarrow B - \lambda_1 I = \begin{pmatrix} 6-4+i & -5 \\ 1 & 2-4+i \end{pmatrix} \Rightarrow v_{11} + (-2+i) v_{22} = 0$

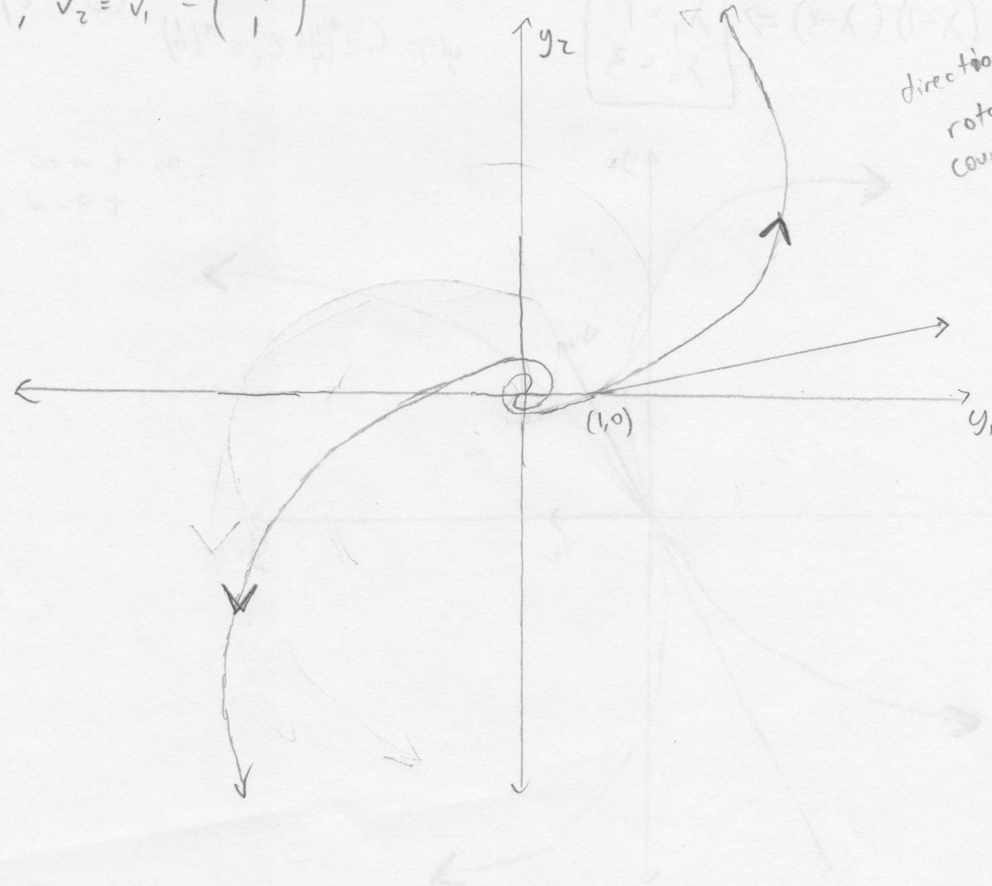
$v_{11} = (2-i) \quad v_{22} = 1$

$v_1 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

for $\lambda_2 = \bar{\lambda}_1$, $v_2 = \bar{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

get direction by trying at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$



direction of rotation: counter-clockwise

(c) $\boxed{\text{spiral source}}$ because $T^2 - 4D = -4 < 0$ and $T = 8 > 0$ and $D = 17 > 0$

Q4 for $y' = \underbrace{\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix}}_{B} y$

(a) $T(B) = 8$

$D(B) = 12 + 5 = 17$

$\boxed{p(\lambda) = \lambda^2 - 8\lambda + 17}$

$\lambda = \dfrac{8 \pm \sqrt{64-68}}{2} = 4 \pm \dfrac{\sqrt{-4}}{2} = 4 \pm i$

$\boxed{\begin{matrix} \lambda_1 = 4 - i \\ \lambda_2 = 4 + i \end{matrix}}$

(b) $\lambda_1 = 4 - i \Rightarrow B - \lambda_1 I = \begin{pmatrix} 6-4+i & -5 \\ 1 & 2-4+i \end{pmatrix} \Rightarrow$ $v_{11} + (-2+i) v_{22} = 0$

$v_{11} = (2-i)$ $v_{22} = 1$

$v_1 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

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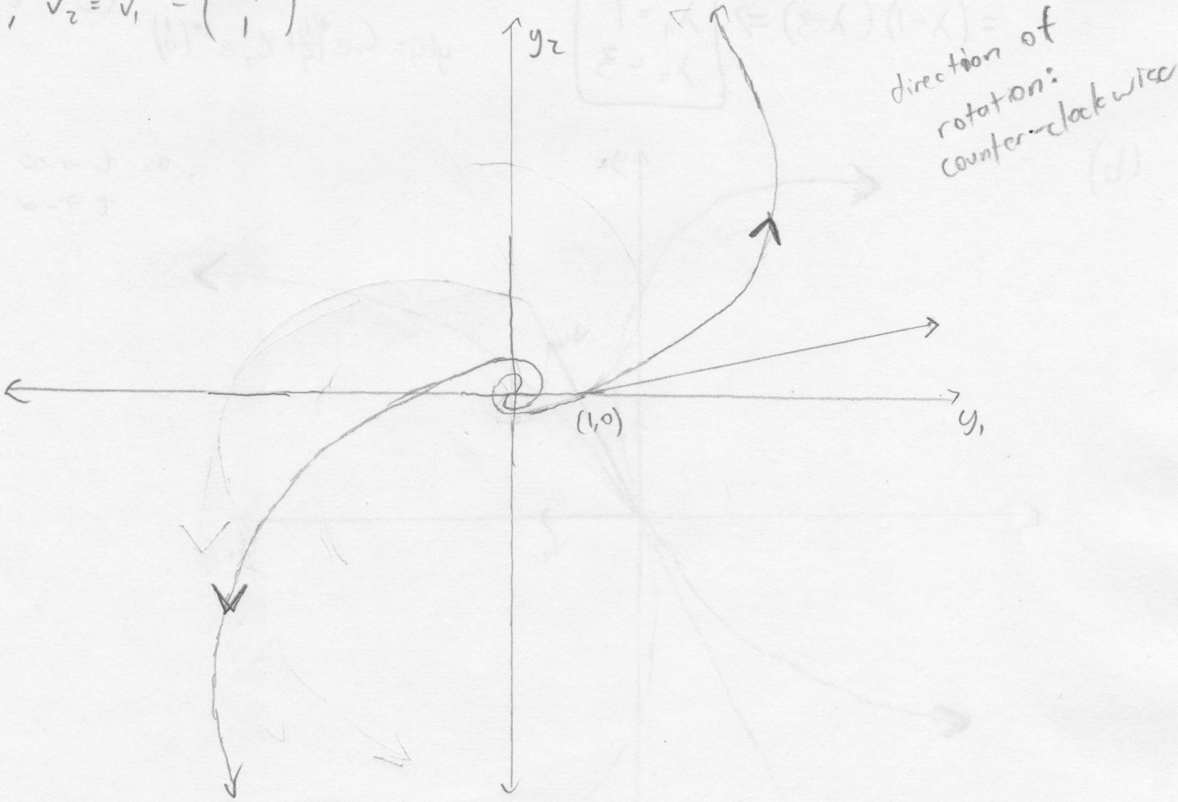

(c) $\boxed{\text{spiral source}}$ because $T^2 - 4D = -4 < 0$ and

$T = 8 > 0$ and $D = 17 > 0$

Q4 for $y' = \underbrace{\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix}}_{B} y$

(a) $T(B) = 8$

$D(B) = 12 + 5 = 17$

$\boxed{p(\lambda) = \lambda^2 - 8\lambda + 17}$

$\lambda = \dfrac{8 \pm \sqrt{64-68}}{2} = 4 \pm \dfrac{\sqrt{-4}}{2} = 4 \pm i$

$\boxed{\begin{aligned} \lambda_1 &= 4 - i \\ \lambda_2 &= 4 + i \end{aligned}}$

(b) $\lambda_1 = 4 - i \Rightarrow B - \lambda_1 I = \begin{pmatrix} 6-4+i & -5 \\ 1 & 2-4+i \end{pmatrix} \Rightarrow$ $v_{11} + (-2+i) v_{22} = 0$

$v_{11} = (2-i) \quad v_{22} = 1$

$v_1 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

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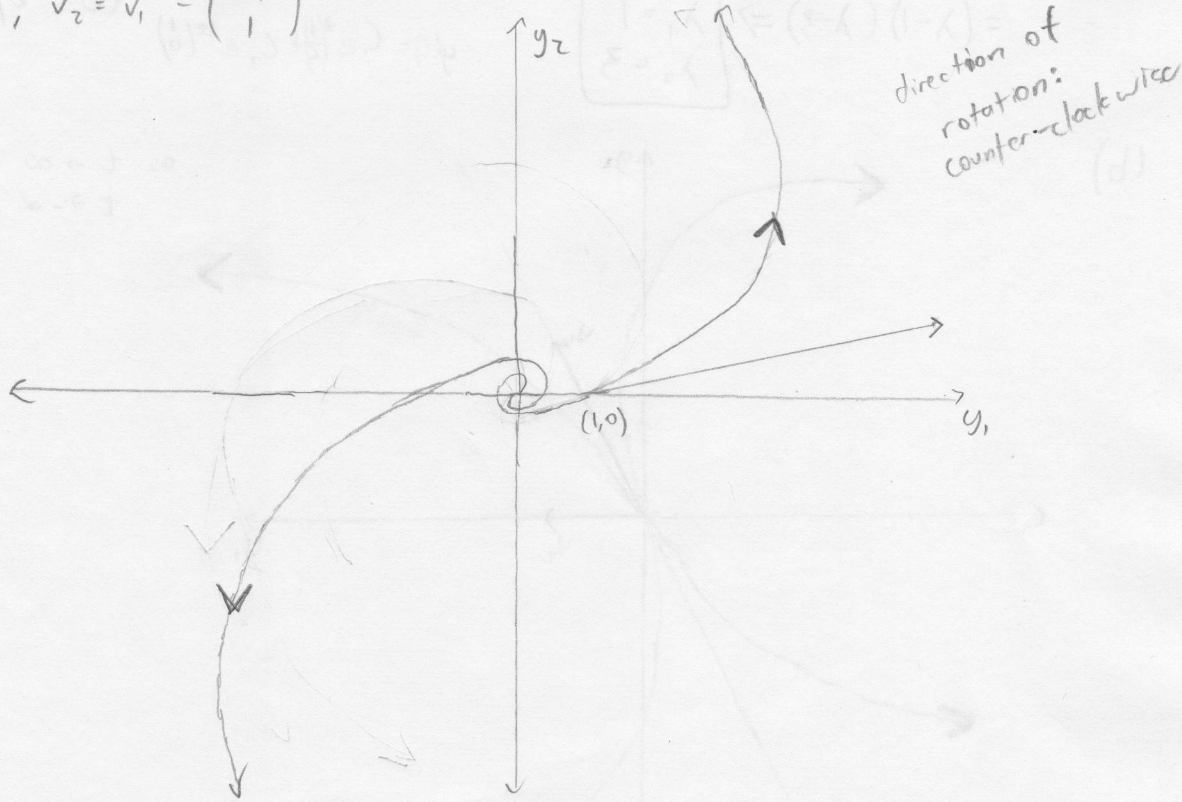
get direction by trying at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$



(c) $\boxed{\text{spiral source}}$ because $T^2 - 4D = -4 < 0$ and

$T = 8 > 0$ and $D = 17 > 0$

5. Let c be a real number and consider the system

$$y' = \begin{pmatrix} c & 0 \\ 1 & 1 \end{pmatrix} y$$

(a) (4 points) For what values of c will the equilibrium solution be a saddle point?

(b) (4 points) For what values of c will the equilibrium solution be a nodal source?

(c) (4 points) What values of c give *nongeneric* equilibrium solutions?

(d) (4 points) Explain why no value of c will give a nodal or spiral sinks.

$T = c+1 \quad D = c$

(a) saddle point where $D<0$

so

saddle point where $c<0$

(b) nodal source where $T^2 - 4D > 0$, $T>0$, and $D>0$

$(c+1)^2 - 4c > 0$ and $c+1>0$ and $c>0$

$c^2+2c+1-4c>0$

$c^2-2c+1>0$

$(c-1)^2>0$ → $c \neq 1$

$c>0$

$c>0$ and $c\neq 1$

nodal source where c is a positive number not equal to 1

(c) nongeneric where $\det(A)=0$ or $T^2=4D$ or $(T(A)=0$ and $D>0)$

$\det(A)=0$ → $c=0$

$T^2=4D$ → $(c+1)^2=4c$, $c^2-2c+1=0$, $(c-1)^2=0$, $c=1$

$T(A)=0$ and $D>0$ → $c+1=0$ and $c>0$, $c=-1$ and $c>0$ → never true

$c=0$ or $c=1$ gives nongeneric equilibrium solutions

(d) To have nodal or spiral sinks, it must be true that $\det(A)>0$ and $T(A)<0$, which means $c>0$ and $c+1<0$, or $c>0$ and $c<-1<0$. Since c cannot be simultaneously greater than 0 and less than 0, it is not possible that a value of c will give nodal or spiral sinks

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$$y' = \begin{pmatrix} c & 0 \\ 1 & 1 \end{pmatrix} y$$

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6. Consider the system

$$y' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y$$

(a) (3 points) Show that the system has infinitely many equilibrium (i.e. constant) solutions.

(b) (3 points) What is the dimension of the nullspace of A?

(c) (4 points) Find a fundamental set of solutions (write your final answers explicitly, without exponential matrices).

$p(\lambda) = \lambda^2 \Rightarrow \lambda = 0$

$\lambda = 0 \rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

part (a) answer here:

let $y = \begin{pmatrix} c \\ c \end{pmatrix}$ where c is any constant

$y' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} c-c \\ c-c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = y'$

since $y = \begin{pmatrix} c \\ c \end{pmatrix}$ is a solution to the system for any constant c, and there are infinitely many possible constants c, the system has infinitely many equilibrium solutions.

(b) dim(nullspace(A)) = # columns − rank(A)

since rows 1 and 2 are the same, $\text{rref}(A) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

so rank(A) = 1

dim(nullspace(A)) = 2 − 1 = **1** is the dimension of the nullspace of A

(c) $T(A) = 0$

$D(A) = -1+1 = 0 \Rightarrow p(\lambda) = \lambda^2 \Rightarrow \lambda = 0$

for $\lambda = 0$, $A - 0I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow y_1 = e^{0t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A-0I)^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \dim(\text{nullity}((A-0I)^2)) = 2$ ✓

$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow y_2 = (v_2 + t(A-0I)v_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t+1 \\ t \end{pmatrix}$

Check for linear independence

$\det\begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} = t - t - 1 = 1 \Rightarrow$ since det of matrix formed by $y_1, y_2 \neq 0$, y_1 and y_2 are linearly independent, so $y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $y_2 = \begin{pmatrix} t+1 \\ t \end{pmatrix}$ form the fundamental set of solutions for $y' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y$

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(b) $\dim(\text{nullspace}(A)) = \#\text{columns} - \text{rank}(A)$

since rows 1 and 2 are the same, $\text{rref}(A) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

so $\text{rank}(A) = 1$

$\dim(\text{nullspace}(A)) = 2 - 1 = \boxed{1}$ is the dimension of the nullspace of A

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$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow y_2 = (v_2 + t(A-0I)v_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t+1 \\ t \end{pmatrix}$

Check for linear independence

$\det\begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} = t - t - 1 = 1 \Rightarrow$ since det of matrix formed by y_1, $y_2 \neq 0$, y_1 and y_2 are linearly independent, so $y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $y_2 = \begin{pmatrix} t+1 \\ t \end{pmatrix}$ form the fundamental set of solutions for $y' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y$

7. Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) (4 points) Give the characteristic polynomial of A and calculate the eigenvalues of A.

(b) (4 points) Determine the dimension of the nullspace of $(A - \lambda_i I)$ for each eigenvalue λ_i found above.

(c) (6 points) Find a fundamental set of solutions to the system $y' = Ay$ (write your final answers explicitly, without exponential matrices)

(a) $\rho(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = (1-\lambda)((-1-\lambda)(-\lambda) - 0) - 1(-1(-\lambda) - 0) + 1(0+0)$

$= (1-\lambda)(\lambda + \lambda^2) - \lambda$

$= \lambda + \lambda^2 - \lambda^2 - \lambda^3 - \lambda$

$\boxed{\rho(\lambda) = -\lambda^3}$

$\boxed{\lambda = 0}$

(b) $A - \lambda I = A - 0I = A$

$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \qquad \begin{matrix} v_1 + v_2 + v_3 = 0 \\ -v_1 - v_2 = 0 \end{matrix} \quad \rightarrow \quad v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

rank-nullity theorem: rank(A) + nullity(A) = 3

$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(A) = 2$, so nullity (dimension of nullspace) of $A = 1$

since $A = A - \lambda I$ for $\lambda = 0$,

$\boxed{\text{dimension of nullspace of } (A - \lambda I) \text{ is } 1}$

7. Consider the following matrix

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(a) $p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = (1-\lambda)((-1-\lambda)(-\lambda) - 0) - 1(-1(-\lambda)-0) + 1(0+0)$

$= (1-\lambda)(\lambda+\lambda^2) - \lambda$

$= \lambda + \lambda^2 - \lambda^2 - \lambda^3 - \lambda$

$\boxed{p(\lambda) = -\lambda^3}$

$\boxed{\lambda = 0}$

(b) $A - \lambda I = A - 0I = A$

$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \quad \begin{matrix} v_1 + v_2 + v_3 = 0 \\ -v_2 - v_2 = 0 \end{matrix} \rightarrow v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

rank-nullity theorem: rank(A) + nullity$(A) = 3$

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since $A = A - \lambda I$ for $\lambda = 0$,

$\boxed{\text{dimension of nullspace of } (A-\lambda I) \text{ is } 1.}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$y_1(t) = e^{0t}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ → need to find p where dimension of nullspace of $(A-0I)^p$ is 3

$$(A-0I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-1 & 1-1 & 1 \\ -1+1 & -1+1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

find v_2 where $(A-0I)^2 v_2 = 0$

$v_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\\0\end{pmatrix}$

rank $(A-0I)^2 = 1$

nullity of $(A-0I)^2 = 2 \neq 3$

keep looking

$$(A-0I)^3 = AA^2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-1 & -1+1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rank$(A^3) = 0$ → nullity $(A^3) = 3$

- to choose v_2 and v_3 in nullspace of $A^3 = (A-0I)^3$, they must be linearly independent of $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, so we choose $v_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$y_2(t) = e^{\lambda t}\left(v_2 + t(A-0I)v_2 + \frac{t^2}{2}(A-0I)^2 v_2\right)$$
$$= e^0\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$
$$= \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}t+1\\-t\\0\end{pmatrix} = y_2(t)$$

$$y_3(t) = e^{\lambda t}\left(v_3 + t(A-0I)v_3 + \frac{t^2}{2}(A-0I)^2 v_3\right)$$
$$= e^0\left(\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\right)$$
$$= \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}t\\0\\0\end{pmatrix} + \begin{pmatrix}t^2/2\\-t^2/2\\0\end{pmatrix}$$
$$y_3(t) = \begin{pmatrix}t^2/2 + t\\-t^2/2\\1\end{pmatrix}$$

check for linear independence

matrix formed by y_1, y_2, y_3

$$\begin{pmatrix} 1 & t+1 & t^2/2+t \\ -1 & -t & -t^2/2 \\ 0 & 0 & 1 \end{pmatrix}$$

determinant is:

$1(-t(1)) - (t+1)(-1+0) + (t^2/2+t)(0+0)$

$= -t + t + 1 = 1 \neq 0$

since det of matrix formed by $y_1, y_2, y_3 \neq 0$, y_1, y_2 and y_3 are linearly independent, so they can form fundamental solution set

fundamental set of solutions for $y' = Ay$ is formed by: $y_1(t) = \begin{pmatrix}1\\-1\\0\end{pmatrix}$, $y_2(t) = \begin{pmatrix}t+1\\-t\\0\end{pmatrix}$, $y_3(t) = \begin{pmatrix}\frac{t^2}{2}+t\\-\frac{t^2}{2}\\1\end{pmatrix}$

8. (20 points) True or False (2 points each). All matrices are assumed to have real entries.

(a) (**True or False**) The differential equation $P(x, y)dx - Q(x, y)dy = 0$ is exact if

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

(b) (**True or False**) If the eigenvalues of a 2×2 matrix A are both positive, then the nonequilibrium solutions to $x' = Ax$ all satisfy $\lim_{t\to\infty} x(t) = 0$.

(this implies different) nodal source

(c) (**True or False**) If x_0 is an equilibrium solution for an autonomous equation $x' = f(x)$, then x_0 is stable if $f'(x_0) > 0$.

unstable Theorem 9.8

(d) (**True or False**) A 3×3 matrix cannot have all complex eigenvalues.

(e) (**True or False**) The function $v(t) = e^{-\int p(t)dt}$ is an integrating factor for

should be −

$$\frac{dx}{dt} + p(t)x = f(t)$$

(f) (**True or False**) There is a polynomial solution to $y' = 4 - y^2$.

$\frac{dy}{dt} = 4 - y^2$

(g) (**True or False**) If A is any $n \times n$ matrix and v is any n-vector, then $y(t) = e^{tA}v$ is a solution to the linear system $y' = Ay$.

(h) (**True or False**) If A, B are 2×2 matrices with $\det(A) = -\det(B)$ and $\text{trace}(A) = -\text{trace}(B)$, then either $x' = Ax$ or $x' = Bx$ has a saddle point equilibrium solution.

if all zeros   Det = 0 $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}$

(i) (**True or False**) If A is a 2×2 matrix with $\det(A) = 0$, then $x' = Ax$ has infinitely many equilibrium solutions

$\lambda^2 - 5\lambda + 0$

$\lambda^2 + 5\lambda + 0$

$\lambda(\lambda + 5)$

(j) (**True or False**) The level set $xy + y^2 = c$ solves the differential equation

$$\underbrace{y}_{P}dx + \underbrace{(2y + x)}_{Q}dy = 0$$

$\frac{dP}{dy} = 1 \quad \frac{dQ}{dx} = 1$

$x_1' = 0 \quad -1 \quad 1$

$x_2' = 0 \quad -2 \quad 2$

$f(x,y) = \int y\,dx = yx + \phi(y)$

$\frac{d}{dx}(xy + y^2)y$

$= y^2 + 2y + x\,dy$

$\frac{dP}{dy} = 1 \quad \frac{dQ}{dx} = 1$ ✓

$\frac{d}{dy}(yx + \phi y) = 2y + x$

$x + \phi'(y) = 2y + x$

$\phi(y) = y^2 \Rightarrow \boxed{xy + y^2}$