1. (5 points) Suppose the first order linear differential equation $x' + p(t)x = f(t)$ has an integrating factor $v(t) = \sec(t)$. What is $p(t)$?

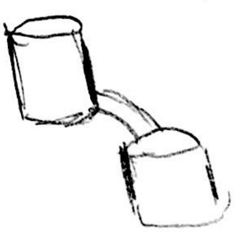
$v(t) = e^{\int p(t)dt}$

$e^{\int p(t)dt} = \sec(t)$

$\int p(t)dt = \ln(\sec(t))$

$\boxed{p(t) = \tan(t)}$

2. (5 points) A recipe calls for a gallon of salt water solution containing 5 ounces of a salt. In your 5 gallon container, you accidentally create a solution with twice that concentration. Suppose you can pour pure water into your container at a rate of 1 gallon per minute while draining it at the same rate. How long will you have to do this to bring your container to the correct concentration? (You may use a calculator to round your answer to one decimal place).



$C_{desired} = \frac{5\,oz}{1\,gal} = 5\,oz/gal$

$C_0 = 10\,oz/gal$

$V_0 = 5\,gal$

$x_0 = 50\,oz$ $\quad x_{desired} = 25\,oz$

rate in = volume in × concentration in = 0 oz/gal

rate out = volume out × concentration out = $1\,gal \times \frac{x(t)}{5} = \frac{x}{5}$ oz/gal

$x'(t) = -\frac{x}{5}$

$\frac{dx}{dt} = -\frac{x}{5}$

$\int \frac{1}{x}dx = \int -\frac{1}{5}dt$

$\ln(x) = -\frac{1}{5}t + C$

$x(t) = Ce^{-\frac{1}{5}t}$

$x(0) = 50$

$50 = C$

$x(t) = 50e^{-\frac{1}{5}t}$

$25 = 50e^{-\frac{1}{5}t}$

$e^{-\frac{1}{5}t} = \frac{1}{2}$

$-\frac{1}{5}t = \ln(\frac{1}{2})$

$\boxed{t = 5\ln(2)\,min \approx 3.5\,min}$

3. (a) (5 points) Consider the differential equation

$$y'' + 2y' - y = -5\sin(2t)$$

Let $y_p(t) = A\cos(2t) + B\sin(2t)$ and use the method of undetermined coefficients to calculate $A$ and $B$. State the particular solution.

$y_p(t) = A\cos(2t) + B\sin(2t)$

$y'_p(t) = 2(-A\sin(2t) + B\cos(2t))$

$y''_p(t) = 4(-A\cos(2t) - B\sin(2t))$

$-4(A\cos(2t)+B\sin(2t)) + 4(-A\sin(2t)+B\cos(2t)) - (A\cos(2t)+B\sin(2t)) = -5\sin(2t)$

$\cos(2t)(-4A+4B-A) + \sin(2t)(-4B-4A-B) = -5\sin(2t)$

$-5A+4B=0$

$-4A-5B=-5$

$-25A+20B=0$

$-16A-20B=-20$

$-41A=-20$

$A=\frac{20}{41}$ $\quad 4B=\frac{100}{41}$

$B=\frac{25}{41}$

$\boxed{y_p(t) = \frac{20}{41}\cos(2t) + \frac{25}{41}\sin(2t)}$

(b) (5 points) Consider the differential equation

$$z'' + 2z' - z = -5e^{2it}$$

Let $z_p = ae^{2it}$ and use the method of undetermined coefficients to calculate $a$ (simplify so that $a$ has real denominator, if necessary). State the (complex) particular solution in the form $x(t) + iy(t)$.

$z_p = ae^{2it}$

$z'_p = 2aie^{2it}$

$z''_p = -4ae^{2it}$

$z''+2z'-z = -5e^{2it}$

$ae^{2it}(-4+4i-1) = -5e^{2it}$

$(-5+4i)a = -5$

$a = \frac{-5}{-5+4i}\cdot\frac{(-5-4i)}{(-5-4i)} = \frac{25+20i}{25+16} = \frac{25+20i}{41}$

$z_p = \left(\frac{(25+20i)}{41}\right)e^{2it} = \left(\frac{(25+20i)}{41}\right)(\cos(2t)+i\sin(2t))$

$z_p = \left(\frac{25}{41}\cos(2t) - \frac{20}{41}\sin(2t)\right) + i\left(\frac{20}{41}\cos(2t) + \frac{25}{41}\sin(2t)\right)$

$x(t)$ $\qquad\qquad$ $y(t)$

4. (20 points) For each the systems below:

$$x' = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} x, \qquad y' = \begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix} y$$

(a) (4 points) Give the characteristic polynomial and calculate the eigenvalues of the matrix.

(b) (4 points) Draw the phase plane portrait . Label the direction of motion (forward time), all eigenvectors, and the direction of rotation, if applicable.

(c) (2 point) Classify the equilibrium solution.

(a) $\det\begin{pmatrix} 3-\lambda & -1 \\ 0 & 1-\lambda \end{pmatrix} = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-1)(\lambda-3) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 3$

$\det\begin{pmatrix} 6-\lambda & -5 \\ 1 & 2-\lambda \end{pmatrix} = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$

$\lambda_1 = 4+i \quad \lambda_2 = 4-i$

(b) $(A-I)v_1 = 0 \qquad (A-3I)v_2 = 0$

$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} v_1 = 0 \qquad \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} v_2 = 0$

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\det\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = -2 \neq 0$

$y(t) = c_1 e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T = 4 > 0$

$D = 3 > 0$

$T^2 - 4D = 16 - 12 = 4 > 0$

Nodal source

$(A-(4+i))v_1 = 0 \qquad (A-(4-i))v_2 = 0$

$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} v_1 = 0 \qquad \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} v_2 = 0$

$v_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

$z(t) = e^{(4+i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = e^{4t}[\cos(t) + i\sin t]\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$= e^{4t}\left[\left(\cos t\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + i\left(\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)\right]$

$= e^{4t}\left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}\right]$

$y(t) = e^{4t}\left[c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}\right]$

$T = 8 > 0$

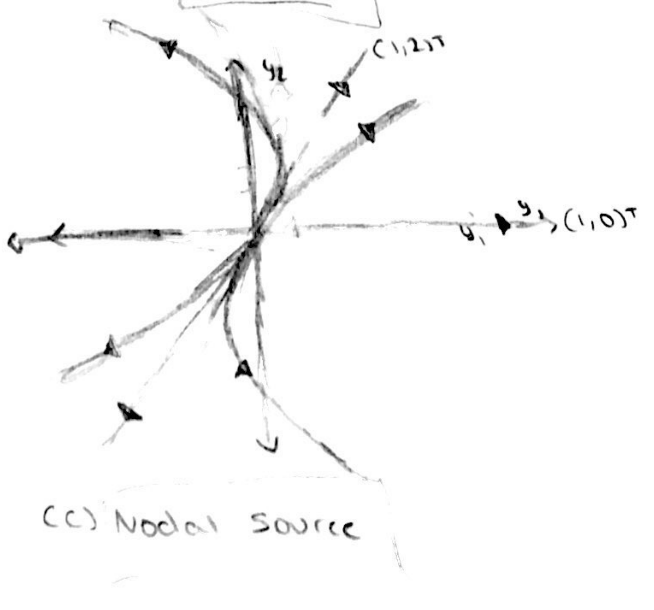
$D = 17 > 0$

$T^2 - 4D = 64 - 68 = -4 < 0$

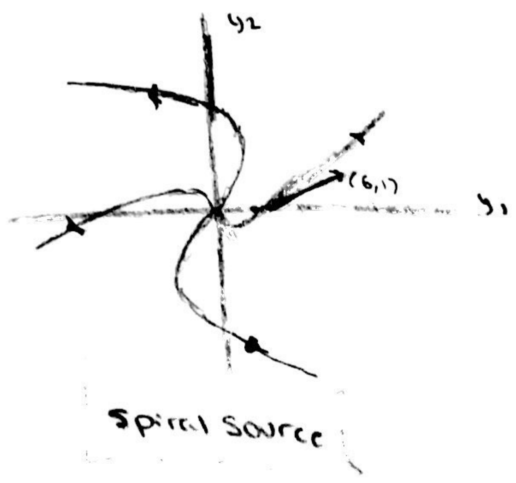
Spiral source

$\begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

Counterclockwise




(c) Nodal Source




Spiral Source

5. Let $c$ be a real number and consider the system

$$y' = \begin{pmatrix} c & 0 \\ 1 & 1 \end{pmatrix} y$$

(a) (4 points) For what values of $c$ will the equilibrium solution be a saddle point?

(b) (4 points) For what values of $c$ will the equilibrium solution be a nodal source?

(c) (4 points) What values of $c$ give *nongeneric* equilibrium solutions?

(d) (4 points) Explain why no value of $c$ will give a nodal or spiral sinks.

(a) $T = c+1 \quad D = c$

saddle point

$T^2 - 4D > 0 \quad D < 0$

$(c+1)^2 - 4c > 0$

$c^2 - 2c + 1 > 0$

$(c-1)^2 > 0$

$c \neq 1$

$D < 0$

$\boxed{c < 0}$

(b)

Nodal source

$T > 0 \quad D > 0$

$T^2 - 4D > 0$

$(c+1)^2 - 4c > 0$

$(c-1)^2 > 0$

$c \neq 1$

$T > 0$

$c + 1 > 0$

$c > -1$

$D > 0$

$c > 0$

$c > 0$ and $c \neq 1$

(c) Nongeneric

$T^2 - 4D = 0$ or $T = 0$ or $D = 0$

$(c+1)^2 - 4c = 0$

$(c-1)^2 = 0$

$\boxed{c = 1}$

$\boxed{c = 0}$

$T = 0$

$c + 1 = 0$

$\cancel{c = -1}$

(d) sinks

$T < 0 \quad D > 0$

$T < 0$

$c + 1 < 0$

$c < -1$

$D > 0$

$c > 0$

Nodal and spiral sinks require that the trace be negative while the determinant is positive. Since $D = c$ and $T = c+1$ no value of $c$ satisfies both condition

6. Consider the system

$$y' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y$$

(a) (3 points) Show that the system has infinitely many equilibrium (i.e. constant) solutions.

(b) (3 points) What is the dimension of the nullspace of $A$?

(c) (4 points) Find a fundamental set of solutions (write your final answers explicitly, without exponential matrices).

(a) $\begin{pmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{pmatrix} = 0$

$\lambda^2 - 1 + 1 = 0$

$\lambda^2 = 0$

$\lambda = 0$

$\det \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0$

Since the determinant is 0, $y' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} y$ has infinitely many equilibrium solutions and all eigenvalues are linearly dependent

(b) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} v_1 = 0$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The dimension of the null space of A is 1

(c) $y_1(t) = e^{0t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\boxed{y_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$ independent

$y_2(t) = e^{0t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t+1 \\ t \end{pmatrix}$

$\boxed{y_2(t) = \begin{pmatrix} t+1 \\ t \end{pmatrix}}$

7. Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) (4 points) Give the characteristic polynomial of $A$ and calculate the eigenvalues of $A$.

(b) (4 points) Determine the dimension of the nullspace of $(A - \lambda_i I)$ for each eigenvalue $\lambda_i$ found above.

(c) (6 points) Find a fundamental set of solutions to the system $y' = Ay$ (write your final answers explicitly, without exponential matrices)

(a) $\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = 0$

$-\lambda(1-\lambda)(-1-\lambda)+\lambda = 0$

$\lambda(\lambda^2-1+1)=0$

$\lambda^3 = 0$

$\lambda = 0$

(b) $A - 0I = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_1 = 0$

$\begin{pmatrix} x_1+x_2+x_3 \\ -x_1-x_2 \\ 0 \end{pmatrix} = 0$

$x_3 = 0$

$x_1 = -x_2$

$\boxed{v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$

dimension of the nullspace of $(A-0I)$ is $\boxed{1}$

$e^{\lambda t} = e^0 = 1$

(c) $[A-0I]v_1 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$

$\boxed{y_1(t) = e^{tA} v_1 = e^{\lambda t} v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$

$[A-0I]^2 v_2 = 0$

$[A-0I]^2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} v_2 = 0$

$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$Av_2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\boxed{y_2(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} t+1 \\ -t \\ 0 \end{pmatrix}}$

$[A-0I]^3 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_3 = 0$

$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$Av_3 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A^2 v_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\det\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 0$ independent

$y_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\boxed{y_3(t) = \begin{pmatrix} \frac{1}{2}t^2 + t \\ -\frac{1}{2}t^2 \\ 1 \end{pmatrix}}$

8. (20 points) True or False (2 points each). All matrices are assumed to have real entries.

(a) (**True** or **False**) The differential equation $P(x, y)dx - Q(x, y)dy = 0$ is exact if

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

(b) (**True** or **False**) If the eigenvalues of a $2 \times 2$ matrix $A$ are both positive, then the nonequilibrium solutions to $x' = Ax$ all satisfy $\lim_{t\to\infty} x(t) = 0$.

(c) (**True** or **False**) If $x_0$ is an equilibrium solution for an autonomous equation $x' = f(x)$, ~~then $x_0$~~ is stable if $f'(x_0) > 0$.

(d) (**True** or **False**) A $3 \times 3$ matrix cannot have all complex eigenvalues.

(e) (**True** or **False**) The function $v(t) = e^{-\int p(t)dt}$ is an integrating factor for

$$\frac{dx}{dt} + p(t)x = f(t)$$

$x' - (-p(t))x = f(t)$

(f) (**True** or **False**) There is a polynomial solution to $y' = 4 - y^2$.

(g) (**True** or **False**) If $A$ is any $n \times n$ matrix and $v$ is any $n$-vector, then $y(t) = e^{tA}v$ is a solution to the linear system $y' = Ay$.

(h) (**True** or **False**) If $A, B$ are $2 \times 2$ matrices with $\det(A) = -\det(B)$ and $\text{trace}(A) = -\text{trace}(B)$, then either $x' = Ax$ or $x' = Bx$ has a saddle point equilibrium solution.

(i) (**True** or **False**) If $A$ is a $2 \times 2$ matrix with $\det(A) = 0$, then $x' = Ax$ has infinitely many equilibrium solutions

(j) (**True** or **False**) The level set $xy + y^2 = c$ solves the differential equation

$$ydx + (2y + x)dy = 0$$

$ydx + xdy + 2ydy = 0$

$ydx + (x+2y)dy = 0$

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