MATH 33B | Midterm 1 | Oct. 22, 2018

STUDENT NAME:

STUDENT ID NUMB

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed.

For instructor use only

Page	Points	Score
2	4	4
3	10	10
4	10	9
5	10	10
6	6	6
Total:	40	39

1. [4 pts] Find the general solution to the differential equation

$$y' = \frac{x + \cos x}{1 + 2y}.$$

You may leave your answer in implicit form.

$$\frac{dy}{dx} = \frac{x + \cos x}{1 + 2y}$$

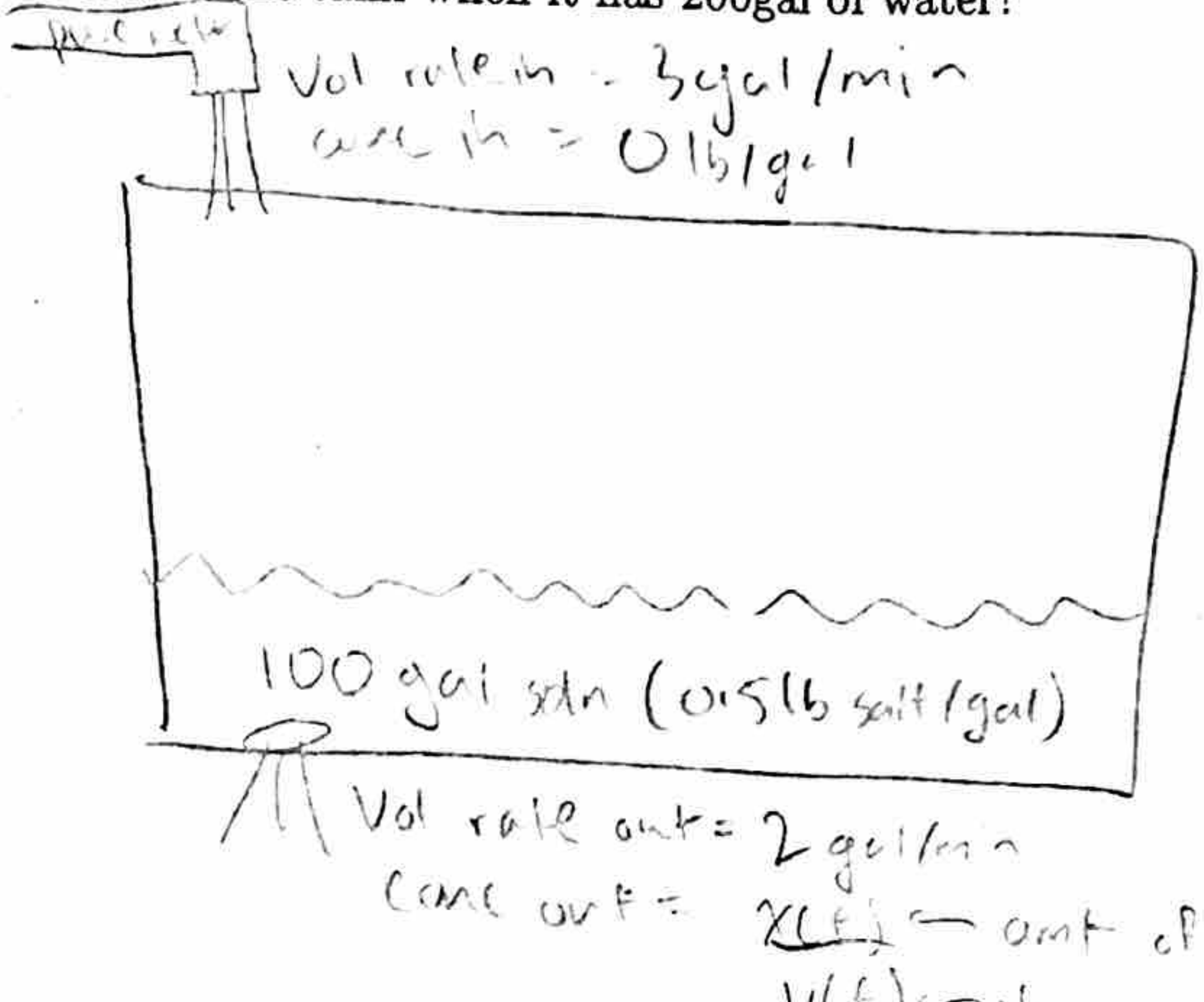
$$dy(1+2y) = (x + \cos x)\,dx$$

$$\int 1 + 2y \cdot dy = \int x + \cos x \, dx$$

$$y + \frac{\cancel{2}y^2}{\cancel{2}} = \frac{x^2}{2} + \sin x + C$$

$$\boxed{y + y^2 = \frac{x^2}{2} + \sin x + C}$$

2. [10 pts] A tank initially contains 100gal of salt-water containing .5lb of salt per gallon of water. At time zero, pure water is poured in at 3gal/min, while simultaneously a drain is opened at the bottom of the tank allowing the mixed salt-water to leave at 2gal/min. What will be the amount of salt in the tank when it has 200gal of water?



$100\,\text{gal} \cdot \frac{0.5\,\text{lbs salt}}{\text{gal}} = 50$ lbs salt initially

$\frac{dx}{dt}$ = rate in − rate out

$v(t) = 100 + (3-2)t = 100 + t$

rate in = 0

rate out = $2\frac{x(t)}{100+t}$

$\frac{dx}{dt} = 0 - \frac{2x(t)}{100+t}$

$\frac{dx}{dt} = -\frac{2x(t)}{100+t}$

$dx = -\frac{2x}{100+t}\,dt$

$\int \frac{dx}{x} = \int \frac{-2}{100+t}\,dt$

$\ln|x| = -2\int \frac{1}{100+t}\,dt = -2\int \frac{du}{u} = -2\ln|u|$
$= -2\ln|100+t| + C$

$u = 100+t$
$du = +dt$
$+du = dt$

$\ln|x| = -2\ln|100+t| + C$

$e^{-2\ln|100+t|+C} = x$

Let $e^C = D$ → $e^C e^{-2\ln(100+t)}$

$D(100+t)^{-2} = x(t)$

$x(0) = 50 = D(100)^{-2} = \frac{D}{100^2}$

$50(10000) = D$

$500000 = D$

$x(t) = \frac{500000}{(100+t)^2}$

$v(t) = 200 = 100 + t$

$t = 100$

$x(100) = \frac{500000}{(200)^2}$

$= \frac{\cancel{500000}}{\cancel{40000}} = \frac{50}{4}$

$x(100) = \boxed{\frac{25}{2}\text{ lbs salt left}}$

✓

3. In this problem we explore the notion of using an integrating factor $\mu(y)$ that depends on y only in order to make the differential form $P(x,y)dx + Q(x,y)dy$ exact.

(a) [6 pts] Suppose that there exists an integrating factor $\mu(y)$ making the new differential form $\mu(y)P(x,y)dx + \mu(y)Q(x,y)dy$ exact. Prove that we must have

$$\frac{d\mu}{dy} = g\mu$$

where g is the function

$$g(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

To be exact:

$$\frac{\partial}{\partial y}(\mu(y)P(x,y)) = \frac{\partial}{\partial x}(\mu(y)Q(x,y))$$

$$\frac{\partial \mu(y)}{\partial y}P + \mu(y)\frac{\partial P}{\partial y} = \mu(y)\frac{\partial Q}{\partial x} \quad \text{why? b/c } \partial\mu/\partial x = 0$$

$$\frac{\partial \mu}{\partial y}P = \mu(y)\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

$$\frac{\partial \mu(y)}{\partial y} = \frac{\partial \mu}{\partial y} = \mu(y)\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = g(y) \checkmark$$

$$\frac{\partial \mu}{\partial y} = \mu\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \checkmark$$

5

(b) [4 pts] Show that defining

$$\mu(y) := e^{\int g(y)dy}$$

satisfies the differential equation $\frac{d\mu}{dy} = g\mu$, and so this should be our definition for $\mu(y)$ in such a case.

$$\frac{d\mu}{\partial y} = \mu'$$

$$\mu' = g\mu$$

$$\int \frac{\mu'}{\mu}dy = \int g\, dy$$

$$\ln|\mu| = \int g\, dy$$

$$\mu(y) = e^{\int g(y)dy} \checkmark$$

(c) [10 pts] Solve the differential equation $(y^2 + 2xy)dx - x^2 dy = 0$, assuming that there is an integrating factor $\mu(y)$ that depends on y only (you may use previous parts of the problem, even if you were not able to prove them).

$\frac{u'}{u} = g$ $P = (y^2 + 2xy)$

$Q = -x^2$

$\frac{\partial P}{\partial y} = 2y + 2x$

$\frac{\partial Q}{\partial x} = -2x$

$g = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{y^2 + 2xy}\left(-2x - 2y - 2x\right)$

$\int g(y) = \int \frac{-4x - 2y}{y^2 + 2xy} dy$ $\frac{-4x - 2y}{y(y+2x)} = \frac{-2(y+2x)}{y(y+2x)} = \frac{-2}{y}$

$\int g(y) = \int \frac{-2}{y} dy$

$\int g(y) dy = -2\int \frac{1}{y} dy = -2\ln(y)$

$u(y) = e^{\ln(y)^{-2}} = \frac{1}{y^2}$

$\frac{y^2 + 2xy}{y^2} dx \quad - \frac{x^2}{y^2} dy$

$\left(1 + \frac{2x}{y}\right)dx - \left(\frac{x^2}{y^2}\right)dy = 0$

$f(x,y) = \int 1 + \frac{2x}{y} dx + \phi(y)$

$\frac{x^2}{y^2}$ $=$

$x + \frac{2}{y} \cdot \frac{x^2}{2} = x + \frac{x^2}{y}$

$f(x,y) = x + \frac{x^2}{y} + \phi(y) = c$

$\frac{\partial f}{\partial y} = \frac{-x^2}{y^2} + \phi'(y) = \frac{-x^2}{y^2}$

$\phi'(y) = 0$

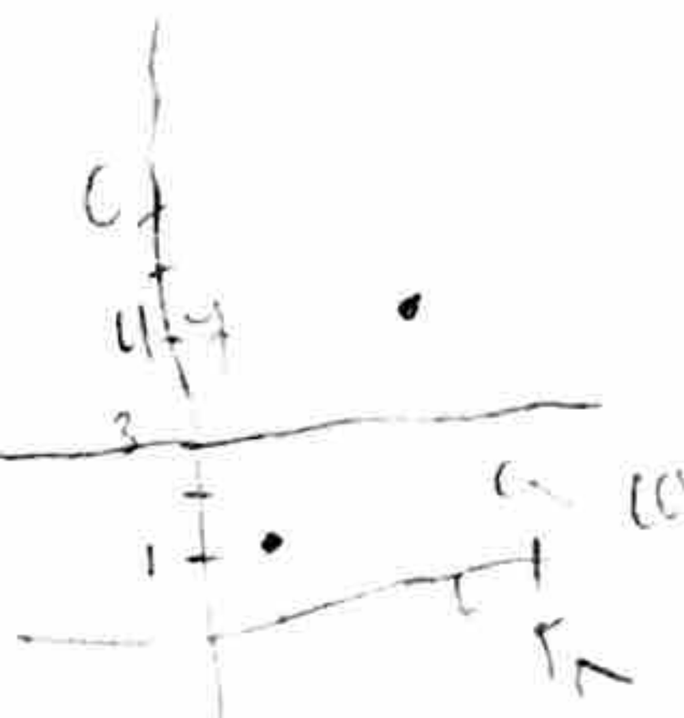
$\phi(y) = 0$

$f(x,y) = C = \boxed{x + \frac{x^2}{y} = C}$

4. Suppose that y is a solution to the initial value problem

$$(t-5)y' = (y-3)\cos(\cos(ty)), \qquad y(1) = 1$$

and that y is actually defined for all t.



(a) [3 pts] Is it possible that $y(4) = 4$? Explain (quote a theorem from class, or explain why it does not apply).

$(t-5)\frac{dy}{dt} = (y-3)\cos(\cos(ty))$

$y' = \frac{(y-3)\cos(\cos ty)}{t-5} = f(y,t)$

$f(y,t)$ is not cont. @ $t=5$, but exists everywhere else

$\frac{\partial f}{\partial y} = \frac{1}{t-5}\left[\cos(\cos(ty)) + (y-3)(-\sin(\cos ty)(-\sin ty)(t)\right])$

$f(y,t)$ guarantees a unique solution everywhere but not @ $t=5$, since $\frac{\partial f}{\partial y}$ is discont.

By uniqueness theorem, there is a unique solution. There is also a an equilibrium solution $y(t)=3$. So solution passing through $y(1)=1$ or $(1,1)$ can't cross $y(t)=3$ because the theorem guarantees a unique solution. So it is NOT possible for $y(4)=4$.

(b) [3 pts] Is it possible that $y(6) = 6$? Explain (quote a theorem from class, or explain why it does not apply).

unique exists on this interval

Is possible, since we guarantee uniqueness from $t:(-\infty,5)$ @ $t=6$ doesn't have to be unique (can cross $y(t)=3$)

As we found above, $f(y,t)$ is continuous for $y:(-\infty,\infty)$ and $t:(-\infty,5)\cup(5,\infty)$ and the same for $\frac{\partial f}{\partial y}$. So the rectangle R which $y(t)$ exists around $y(1)=1$ is bounded by
$\begin{cases} y:(-\infty,\infty) \\ t:(-\infty,5) \end{cases}$

since interval for t must contain $t=1$. We cannot guarantee existence outside of these bounds by the existence theorem. So it's not possible for $(t=6)$ and $y(6)=6$. Also, by the uniqueness theorem, the solution through the point $(1,1)$ cannot cross the $y(t)=3$ solution since there are unique solns (see part a). So that solution can't

disregard