STUDENT NAME: Zhicheng Ren

STUDENT ID NUMBER: 304972327

1/A

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed.

For instructor use only

Page	Points	Score
2	4	4
3	10	9
4	10	10
5	10	9
6	6	2
Total:	40	34

1. [4 pts] Find the general solution to the differential equation

$$y' = \frac{x + \cos x}{1 + 2y}.$$

You may leave your answer in implicit form.

$$\frac{dy}{dx} = \frac{x + \cos x}{1+2y}$$

$$(1+2y)\,dy = (x+\cos x)\,dx$$

$$\int (1+2y)\,dy = \int (x+\cos x)\,dx$$

$$y + y^2 = \frac{x^2}{2} + \sin x + C$$

2. [10 pts] A tank initially contains 100gal of salt-water containing .5lb of salt per gallon of water. At time zero, pure water is poured in at 3gal/min, while simultaneously a drain is opened at the bottom of the tank allowing the mixed salt-water to leave at 2gal/min. What will be the amount of salt in the tank when it has 200gal of water?

$$\text{rate} = \text{rate in} - \text{rate out}$$
$$= 3 \cdot 0 - 2 \cdot \frac{x}{100+t}$$
$$x' = -\frac{2}{100+t} x$$
$$\frac{dx}{dt} = -\frac{2}{100+t} x$$
$$\frac{1}{x} dx = -\frac{2}{100+t} dt$$
$$\int \frac{1}{x} dx = -\int \frac{2}{100+t} dt$$
$$\ln|x| = -2\ln|100+t| + C$$
$$\ln|x| = \ln(100+t)^{-2} + C$$
$$\because x \geq 0 \quad x = e^{\ln(100+t)^{-2} + C}$$
$$x(t) = \frac{1}{(100+t)^2} \cdot e^C$$
$$\text{when } t = 0 \quad x(t) = 5$$
$$5 = \frac{1}{100^2} \cdot e^C$$
$$50000 = e^C$$
$$C = \ln 50000$$
$$x(t) = \frac{50000}{(100+t)^2}$$
$$200 - 100 = 100$$
$$t = 100$$
$$x(t) = \frac{50000}{200^2} = \frac{50000}{40000} = 1.25 \text{lb}$$

3. In this problem we explore the notion of using an integrating factor $\mu(y)$ that depends on y only in order to make the differential form $P(x,y)dx + Q(x,y)dy$ exact.

(a) [6 pts] Suppose that there exists an integrating factor $\mu(y)$ making the new differential form $\mu(y)P(x,y)dx + \mu(y)Q(x,y)dy$ exact. Prove that we must have

$$\frac{d\mu}{dy} = g\mu \quad (1)$$

where g is the function

$$g(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right). \quad (2)$$

to make the differential form exact

$$\frac{\partial(\mu(y)P(x,y))}{\partial y} - \frac{\partial(\mu(y)Q(x,y))}{\partial x} = 0$$

$$0 = \frac{d}{dy}\mu(y)\,P(x,y) + \mu(y)\frac{\partial P}{\partial y} - \left(0 + \mu(y)\frac{\partial Q}{\partial x}\right)$$

$$\frac{d\mu}{dy}P = \mu(y)\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$\frac{d\mu}{dy} = \frac{\mu(y)}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$\because (1)$, $\therefore g(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \Rightarrow (2)$

(b) [4 pts] Show that defining

$$\mu(y) := e^{\int g(y)dy}$$

satisfies the differential equation $\frac{d\mu}{dy} = g\mu$, and so this should be our definition for $\mu(y)$ in such a case.

$$\frac{d\mu}{dy} = \underbrace{\frac{d}{dy}\left(\int g(y)dy\right)} e^{\int g(y)dy}$$

$$= g(y)\, e^{\int g(y)dy}$$

$$= g(y)\cdot\mu(y)$$

$$= g\mu$$

(c) [10 pts] Solve the differential equation $(y^2 + 2xy)dx - x^2dy = 0$, assuming that there is an integrating factor $\mu(y)$ that depends on y only (you may use previous parts of the problem, even if you were not able to prove them).

$$\frac{\partial Q}{\partial x} = -2x \qquad \frac{\partial P}{\partial y} = 2y + 2x$$

$$g(y) = \frac{1}{y^2 + 2xy}(-2x - 2y - 2x)$$

$$= \frac{-4x - 2y}{y^2 + 2xy} = \frac{-2(y+2x)}{y(y+2x)} = -\frac{2}{y}$$

$$\mu(y) = e^{\int -\frac{2}{y}\,dy}$$

$$= e^{-2\ln y}$$

$$= e^{\ln y^{-2}}$$

$$= \frac{1}{y^2}$$

$$\frac{1}{y^2}(y^2 + 2xy)dx - \frac{1}{y^2}(x^2)dy = 0$$

$$\left(1 + \frac{2x}{y}\right)dx - \left(\frac{x^2}{y^2}\right)dy = 0$$

$$F(x,y) = \int \left(1 + \frac{2x}{y}\right)dx + \phi(y)$$

$$= x + \frac{x^2}{y} + \phi(y)$$

$$\frac{\partial F}{\partial y} = -\frac{x^2}{y^2} + \phi'(y)$$

$$\phi'(y) = 0 \qquad \phi(y) = C$$

$$F(x,y) = x + \frac{x^2}{y} + C$$

-1

4. Suppose that y is a solution to the initial value problem

$$(t-5)y' = (y-3)\cos(\cos(ty)), \qquad y(1) = 1$$

and that y is actually defined for all t.

(a) [3 pts] Is it possible that $y(4) = 4$? Explain (quote a theorem from class, or explain why it does not apply).

$$y' = \frac{y-3}{t-5}\cos(\cos(ty)) \quad \text{let } F = \frac{y-3}{t-5}(\cos(\cos(ty)))$$

$$\frac{\partial F}{\partial t} = -\frac{y-3}{(t-5)^2}(\cos(\cos(ty))) + \frac{y-3}{t-5}\sin(\cos(ty))\sin(ty)\cdot y$$

since both $\frac{\partial F}{\partial t}$ and y' are defined/continuous as long as $t \neq 5$,

by the theorem of uniqueness solution, when $t=4$, $y=4$, it produces a unique general solution which is defined for all t $\Rightarrow$ possible

?

-2

(b) [3 pts] Is it possible that $y(6) = 6$? Explain (quote a theorem from class, or explain why it does not apply).

$$y' = \frac{y-3}{t-5}\cos(\cos(ty))$$

$$\frac{\partial F}{\partial t} = -\frac{y-3}{(t-5)^2}\cdot\cos(\cos(ty)) + \frac{y-3}{t-5}\sin(\cos(ty))\cdot\sin(ty)\cdot y$$

since both $\frac{\partial F}{\partial t}$ and y' are defined/continuous as long as $t \neq 5$, by the theorem of uniqueness solution, when $t=6$, $y=6$, it produces a unique general solution which is defined for all t $\Rightarrow$ possible

?

-2