21W-MATH33B-2 Final Exam

JOE PINTO, SR

TOTAL POINTS

96 / 100

QUESTION 1

Q1 8 pts

1.1 1a 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

1.2 1b 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

1.3 1c 0 / 2

- 0 pts Correct

✓ **- 2 pts** Incorrect

1.4 1d 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

QUESTION 2

2 Q2 5 / 8

- 0 pts Correct

- 1 pts Sign error / doesn't satisfy the initial condition

- 1 pts Misc. error

- 2 pts Didn't write y explicitly

- 2 pts Computational error

- 3 pts Didn't use the initial condition (or applied it in a way that doesn't make sense)

✓ **- 3 pts** Error in part 1 that throws off the rest

- 4 pts Subsequent parts unfinished

- 4 pts No constant in part 2

- 8 pts Incorrect or blank

y is also a function of x, so (xy)' = xy' + y

QUESTION 3

3 Q3 10 / 10

✓ **- 0 pts** Correct

- 2 pts Did not rule out possibility that top entry of an eigenvector is $$0$$

- 1 pts Minor errors

- 3 pts Moderate errors

- 5 pts Did not consider eigenvectors at all

- 5 pts Believed eigenvectors could be chosen at will

- 6 pts Did not consider $$p_1$$ at all, and thus did not consider top row at all

- 5 pts Believed $$p_1$$ solution had eigenvectors, and thus never considered top row

- 6 pts Showed $$p_1=p_M$$, and made no further claims

- 1 pts Correctly found eigenvectors, but did not mention reduction to top row only

- 5 pts Claimed one eigenvector was $$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$, nothing mentioned about the other

- 3 pts Correctly found eigenvectors, but claimed that combining vectors and scalars could reduce to the other solution

- 4 pts Did not find eigenvectors, and claimed that they could be combined with scalars to arrive at $$p_1$$ solution

- 3 pts Eigenvectors would not work if 0 is an eigenvalue, also did not properly indicate top row and absorption of constants

- 4 pts Began work on eigenvectors but could not finish

- 6 pts Believed both versions led to eigenvectors both being $$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- 8 pts Complete misunderstanding of matrix version, or did not attempt matrix version at all

QUESTION 4

Q4 10 pts

4.1 4a **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Incorrect definition of matrix exponential.

4.2 4b **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Incorrect value of $$\alpha$$.

- 0.5 pts Using $$\alpha A$$ instead of $$A$$ in the formula.

- 0.5 pts Matrix computation errors.

QUESTION 5

Q5 46 pts

5.1 5a **3 / 6**

- 0 pts Correct

- 1 pts Minor errors with mass

- 2 pts Errors with signs

- 2 pts Had damping terms for other masses

- 4 pts No attempt at damping terms

✓ **- 3 pts Had damping attached to position instead of velocity**

- 4 pts Had $$c$$ present on all terms?

- 4 pts Had damping listed as a constant, tried to write as inhomogeneous?

- 5 pts Serious mistakes

- 2 pts One damping term incorrect

- 2 pts Attached $$x_1,x_2$$ to $$c$$ in the matrix (but in the correct places)?

- 2 pts Missing one $$\frac{k_2}{m_i}$$ term

- 2 pts Simple mistake with $$x'$$ vs $$y'$$

- 3 pts No mass terms

- 2 pts Included $$x,x',y,y'$$ terms in the matrix

- 2 pts Errors with $$k_i$$'s

5.2 5b **4 / 4**

✓ **- 0 pts** Correct

- 1 pts Believed in a fixed sign for $$\beta$$

- 2 pts Complex, unsure of $$\alpha$$ sign

- 2 pts Claimed $$\alpha>0$$

- 2 pts Claimed $$\alpha=0$$

- 4 pts Nothing helpful

- 2 pts "Complex and negative"

- 1 pts Claimed $$\alpha>0$$ to maintain the minus sign in the quadratic formula

- 3 pts "Small distinct eigenvalues, $$\alpha>0,\beta<0$$"

- 1 pts Claimed $$\alpha\leq 0$$ only

5.3 5c **3 / 4**

- 0 pts Correct

✓ **- 1 pts Implied only two eigenvalues**

- 2 pts Real, but sign wrong (or unmentioned/unclear)

- 4 pts Nothing helpful

- 3 pts Believed complex with negative $$\beta$$

- 2 pts Believed half of the eigenvalues were complex

5.4 5d **4 / 4**

✓ **- 0 pts** Correct

5.5 5e **6 / 6**

✓ **- 0 pts** Correct

- 1 pts Incorrect value of $$(A - \lambda I)^2$$.

5.6 5f **6 / 6**

✓ **- 0 pts** Correct

- 3 pts Checked eigenvalue but not eigenvector

- 3 pts Computed A- lambda I but did not apply to the vector

- 6 pts Incorrect or no work

5.7 5g **2 / 2**

✓ **- 0 pts** Correct

- 2 pts Incorrect / no answer

- 1 pts Click here to replace this description.

5.8 5h **10 / 10**

✓ **- 0 pts** Correct

- 2 pts error in solution associated to generalized eigenvector w

- **3 pts** incorrect solution associated to w
- **5 pts** Does not give real valued solutions associated to complex conjugate pair of eigenvectors
- **1 pts** incorrect solution associated to v
- **2 pts** error in solutions associated to complex conjugate pair
- **4 pts** incorrect/incomplete solutions associated to complex conjugate pair
- **10 pts** Incorrect / no work

5.9 5i **4 / 4**

✓ **- 0 pts** **Correct**

- **4 pts** Incorrect / no work
- **2 pts** Does not indicate long term behavior or incorrect assessment
- **2 pts** Incorrect solutions x(t), y(t)
- **1 pts** Does not indicate long term behavior approaches (0,0)
- **1 pts** error in one of the solutions

QUESTION 6

Q6 18 pts

6.1 6a **4 / 4**

✓ **+ 2 pts** **y = 2**

✓ **+ 2 pts** **$$x = \pi k, k\in\mathbb Z$$**

+ **1 pts** Only finitely many solutions for x
+ **0 pts** Incorrect or blank

6.2 6b **8 / 8**

✓ **+ 4 pts** **Even multiples of $$\pi$$: saddle**

✓ **+ 4 pts** **Odd multiples of $$\pi$$: spiral sink**

+ **0 pts** Incorrect or blank
+ **4 pts** Med. partial credit (e.g. sign error/results backwards)
+ **2 pts** Partial credit
+ **1 pts** Compute Jacobian but no other progress

6.3 6c **6 / 6**

✓ **+ 1 pts** **$$A = 0$$**

✓ **+ 1 pts** **$$A = 1$$**

✓ **+ 1 pts** **$$A = -1$$**

✓ **+ 1 pts** **$$A = 0$$ is inconclusive for any $$k$$**

✓ **+ 1 pts** **$$A = 1$$ is inconclusive for odd $$k$$**

✓ **+ 1 pts** **$$A = -1$$ is inconclusive for even $$k$$**

+ **0 pts** Incorrect or blank
+ **2 pts** Some progress but not specific, or partial progress due to errors in (b)

6.4 6d(bonus) **5 / 0**

✓ **+ 5 pts** **Correct**

+ **3 pts** Right idea, but some issues with explanation
+ **2 pts** Believes non-saddles are always one type of sink
+ **2 pts** Believes non-saddles can be either source or sink
+ **1 pts** Believes non-saddles are always source
+ **1 pts** On the right track, but with some incomplete/incorrect elements

- **0 pts** Nothing helpful

1. Circle TRUE or FALSE for each; no explanation necessary.

(a) [2 pts] The initial value problem

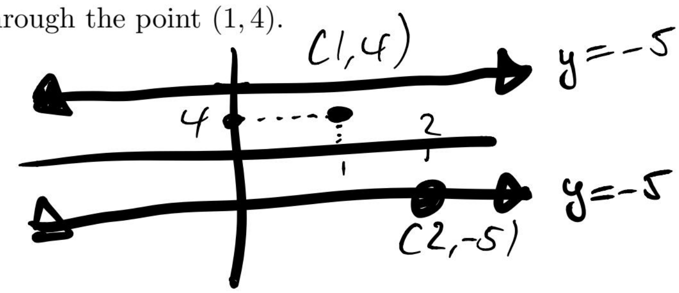
$$x' = |x|, \quad x(0) = 0$$

has no solution because $\frac{d}{dx}|x|$ does not exist at $x = 0$.

TRUE FALSE

(b) [2 pts] Consider the 1st order ODE $y' = \sqrt{y^2 - 25}$. The existence and uniqueness theorem guarantees that there is a unique solution through the point $(1, 4)$.

TRUE FALSE



(c) [2 pts] Consider the 1st order ODE $y' = \sqrt{y^2 - 25}$. The existence and uniqueness theorem guarantees that there is a unique solution through the point $(2, -5)$.

TRUE FALSE

(d) [2 pts] Consider a linear homogeneous nxn system $\overrightarrow{\mathbf{y}}' = A\overrightarrow{\mathbf{y}}$ where A has a real eigenvalue λ that is repeated k times in the characteristic polynomial. Then there will be exactly k linearly independent solutions $\overrightarrow{\mathbf{y}}_1(t), \ldots, \overrightarrow{\mathbf{y}}_k(t)$ corresponding to λ for this system.

TRUE FALSE

1.1 1a **2 / 2**

✓ **- 0 pts** Correct

- 2 pts Incorrect

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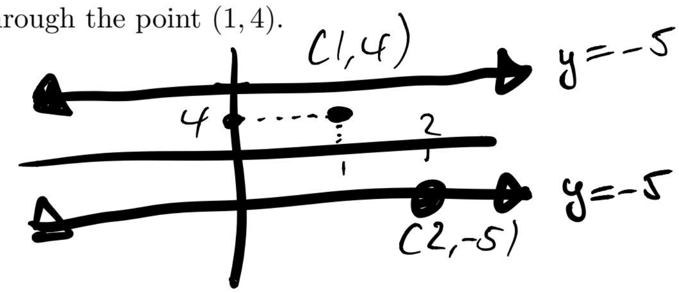

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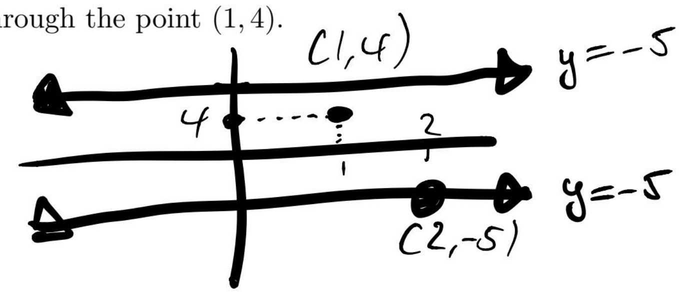

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1.3 1c 0 / 2

- **- 0 pts** Correct
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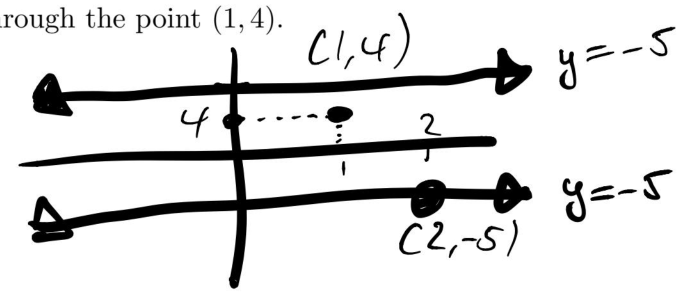

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1.4 1d 2 / 2

✓ - **0 pts** Correct

- **2 pts** Incorrect

2. [8 pts] Consider the first order non-linear differential equation

$$xy' + y = x^4 y^3$$

where y is a function of $x > 0$. Solve for $y(x)$ having initial value $y(1) = -1$ by following these steps:

1. Define a new function $z(x) := xy$, find $z'(x) = \frac{dz}{dx}$, and substitute to turn the given differential equation into a new one for z', z, and x.
2. Solve the new differential equation for $z(x)$ (it should be easier).
3. Knowing $z(x)$, find $y(x)$.

Write your answer as an *explicit* function $y(x)$.

(1) Let $z(x) = xy$, $z'(x) = y \Rightarrow y = z'(x)$

Note $y' = \frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz} = y\frac{dy}{dz} = y\left(\frac{dz}{dy}\right)^{-1} = y(x)^{-1} = \frac{y}{x}$

$$xy' + z' = xz^3$$

$$x\left(\frac{y}{x}\right) + z' = xz^3$$

$$2z' = xz^3$$

$$2\frac{dz}{dx} = xz^3$$

$$\Rightarrow \frac{2}{z^3}\,dz = x\,dx$$

integrating both sides:

$$\int \frac{2}{z^3}\,dz = \int x\,dx$$

$$-z^{-2} + C_0 = \frac{x^2}{2} + C_1$$

$$(C = C_1 - C_0)$$

$$\Rightarrow -\frac{1}{z^2} = \frac{x^2}{2} + C_2$$

so $z^2 = \left(-\frac{x^2}{2} + C\right)^{-1}$ $(C = C_2)$

$$z = \pm\left(-\frac{x^2}{2} + C\right)^{-\frac{1}{2}}$$

but $z = xy$,

so $y = \frac{z}{x} = \pm\frac{\left(-\frac{x^2}{2} + C\right)^{-\frac{1}{2}}}{x}$

$$= \pm\frac{1}{x\sqrt{-\frac{x^2}{2} + C}}$$

since $y(1) = -1$, take -ve soln and solve:

$$-1 = \frac{-1}{1\sqrt{-\frac{1}{2} + C}}$$

$$\sqrt{-\frac{1}{2} + C} = 1$$

$$-\frac{1}{2} + C = 1$$

$$\Rightarrow C = \frac{3}{2}$$

So $y(x) = \frac{-1}{x\sqrt{\frac{3}{2} - \frac{x^2}{2}}}$

2 Q2 **5 / 8**

- **- 0 pts** Correct
- **- 1 pts** Sign error / doesn't satisfy the initial condition
- **- 1 pts** Misc. error
- **- 2 pts** Didn't write y explicitly
- **- 2 pts** Computational error
- **- 3 pts** Didn't use the initial condition (or applied it in a way that doesn't make sense)
- ✓ **- 3 pts Error in part 1 that throws off the rest**
- **- 4 pts** Subsequent parts unfinished
- **- 4 pts** No constant in part 2
- **- 8 pts** Incorrect or blank

y is also a function of x, so (xy)' = xy' + y

3. [10 pts] Suppose we have a second order linear, homogeneous differential equation

$$x'' + ax' + bx = 0.$$

We can solve directly using the characteristic polynomial for this equation (call this polynomial $p_1(\lambda)$) OR we can transform this into a system of 1st order linear equations and solve using the characteristic polynomial for the corresponding matrix (call that polynomial $p_M(\lambda)$, M for matrix).

Prove that these two approaches are equivalent. That is, prove that $p_1(\lambda) = p_M(\lambda)$ AND that our resulting general solution for x is the same. You may assume that the eigenvalues are real and distinct (although this is not necessary).

① $P_1(\lambda) = \lambda^2 + a\lambda + b$ (by definition)

② define $u_1 = x$, $u_2 = x'$ $\Rightarrow$ $u_1' = x' = u_2$, $u_2' = x'' = -a u_2 - b u_1$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$ so $y' = Ay$ for $A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$, $y = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$P_M(\lambda) = \lambda^2 - T\lambda + D$
$= \lambda^2 + a\lambda + b = P_1(\lambda)$

for ①: we know $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

for ②: for λ_1: $\ker(A - \lambda_1 I) = \ker \begin{pmatrix} -\lambda_1 & 1 \\ -b & -a-\lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$

checking: $A\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 + \lambda_1 \\ -(\lambda_1^2 + a\lambda + b) \end{pmatrix} = \begin{pmatrix} 0 \\ -P_M(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0^* \end{pmatrix}$ ✓ (since $-P_M(\lambda) = 0$ by definition)

similarly, for λ_2: $\ker(A - \lambda_1 I) = \ker \begin{pmatrix} -\lambda_2 & 1 \\ -b & -a-\lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

Note that $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ is lin. indep. from $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$. ($\lambda_1 \neq k\lambda_2$)

So $U(t) = C_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + C_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

recall $x(t) = u_1(t) = C_1 e^{\lambda_1 t}(1) + C_2 e^{\lambda_2 t}(1)$
$= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ $\Rightarrow$ Same result as from ①

∎

3 Q3 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 2 pts** Did not rule out possibility that top entry of an eigenvector is $$0$$
- **- 1 pts** Minor errors
- **- 3 pts** Moderate errors
- **- 5 pts** Did not consider eigenvectors at all
- **- 5 pts** Believed eigenvectors could be chosen at will
- **- 6 pts** Did not consider $$p_1$$ at all, and thus did not consider top row at all
- **- 5 pts** Believed $$p_1$$ solution had eigenvectors, and thus never considered top row
- **- 6 pts** Showed $$p_1=p_M$$, and made no further claims
- **- 1 pts** Correctly found eigenvectors, but did not mention reduction to top row only
- **- 5 pts** Claimed one eigenvector was $$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$, nothing mentioned about the other
- **- 3 pts** Correctly found eigenvectors, but claimed that combining vectors and scalars could reduce to the other solution
- **- 4 pts** Did not find eigenvectors, and claimed that they could be combined with scalars to arrive at $$p_1$$ solution
- **- 3 pts** Eigenvectors would not work if 0 is an eigenvalue, also did not properly indicate top row and absorption of constants
- **- 4 pts** Began work on eigenvectors but could not finish
- **- 6 pts** Believed both versions led to eigenvectors both being $$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
- **- 8 pts** Complete misunderstanding of matrix version, or did not attempt matrix version at all

4. Suppose that we have some matrix A who satisfies $A^2 = \alpha A$, where $\alpha \neq 0$ is some scalar.

(a) [5 pts] Using the definition of e^{tA}, prove that

$$e^{tA} = I + \frac{e^{\alpha t} - 1}{\alpha} A.$$

since $A^2 = \alpha A$, $\alpha \neq 0$,

we know $A^3 = AA^2 = A(\alpha A) = \alpha(A^2) = \alpha^2 A$

$A^4 = AA^3 = A(\alpha^2 A) = \alpha^3 A$

...

so $A^k = \alpha^{k-1} A$

so then, using Taylor expansion of e^{tA},

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 \dots$$

$$= I + tA + \frac{t^2}{2!}(\alpha A) + \frac{t^3}{3!}(\alpha^2 A) + \dots$$

$$= I + \left(t + \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^3}{3!} + \dots\right)A \overset{*}{=} I + \left(\frac{e^{\alpha t} - 1}{\alpha}\right)A \quad \square$$

* NOTE:

since $e^{\alpha t} = 1 + \frac{1}{1!}\alpha t + \frac{1}{2!}(\alpha t)^2 + \frac{1}{3!}(\alpha t)^3 \dots$

then $e^{\alpha t} - 1 = t\alpha + \frac{1}{2!}\alpha^2 t^2 + \frac{1}{3!}\alpha^3 t^3 \dots$

$$\Rightarrow \frac{e^{\alpha t} - 1}{\alpha} = t + \frac{1}{2!}\alpha t^2 + \frac{1}{3!}\alpha^2 t^3 \dots$$

(b) [5 pts] Use the previous part (even if you could not show it) to compute e^{tA} for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note:

$$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A \Rightarrow \alpha = 3$$

$$\text{so } e^{tA} = I + \frac{e^{3t} - 1}{3} A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{e^{3t} - 1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} e^{3t}+2 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}+2 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}+2 \end{pmatrix}$$

4.1 4a 5 / 5

- ✓ **- 0 pts** Correct
- **- 1 pts** Incorrect definition of matrix exponential.

4. Suppose that we have some matrix A who satisfies $A^2 = \alpha A$, where $\alpha \neq 0$ is some scalar.

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so $A^k = \alpha^{k-1} A$

so then, using Taylor expansion of e^{tA},

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 \ldots$$

$$= I + tA + \frac{t^2}{2!}(\alpha A) + \frac{t^3}{3!}(\alpha^2 A) + \ldots$$

$$= I + \left(t + \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^3}{3!} + \ldots\right)A \overset{*}{=} I + \left(\frac{e^{\alpha t} - 1}{\alpha}\right)A \quad \square$$

* NOTE:

since $e^{\alpha t} = 1 + \frac{1}{1!}\alpha t + \frac{1}{2!}(\alpha t)^2 + \frac{1}{3!}(\alpha t)^3 \ldots$

then $e^{\alpha t} - 1 = t\alpha + \frac{1}{2!}\alpha^2 t^2 + \frac{1}{3!}\alpha^3 t^3 \ldots$

$\Rightarrow \frac{e^{\alpha t} - 1}{\alpha} = t + \frac{1}{2!}\alpha t^2 + \frac{1}{3!}\alpha^2 t^3 \ldots$

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$$\text{so } e^{tA} = I + \frac{e^{3t} - 1}{3} A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{e^{3t} - 1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} e^{3t}+2 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}+2 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}+2 \end{pmatrix}$$

4.2 4b **5 / 5**

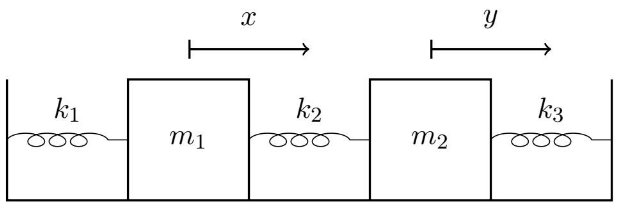
✓ **- 0 pts** Correct

- 1 pts Incorrect value of $$\alpha$$.

- 0.5 pts Using $$\alpha A$$ instead of $$A$$ in the formula.

- 0.5 pts Matrix computation errors.

5. Consider a system of two masses m_1 and m_2 connected by springs with spring constants k_1, k_2, k_3 as shown below, where all of the motion takes place in a liquid with damping constant c. The horizontal motions of m_1 and m_2 are tracked by x and y respectively.



(a) [6 pts] Construct the system of 2nd order differential equations for $x(t)$ and $y(t)$, and convert this system into a system of 1st order linear systems written in matrix format $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$. You can leave the constants $m_1, m_2, k_1, k_2, k_3, c$ in your answer.

Note: total force $= mx''$ or $= my''$ (for mass 1 and 2 respectively)

for mass 1:

$$m_1 x'' = -k_1 x - k_2 x + k_2 y - cx$$

for mass 2:

$$m_2 y'' = k_2 x - k_2 y - k_3 y - cy$$

translate to $\bar{u}' = A\bar{u}$:

$u_1 = x$, $u_2 = x'$, $u_3 = y$, $u_4 = y'$ $\Rightarrow$

$$u_1' = x' = u_2$$

$$u_2' = x'' = -\frac{k_1}{m_1}x - \frac{k_2}{m_1}x + \frac{k_2}{m_1}y - \frac{c}{m_1}x = -\frac{k_1}{m_1}u_1 - \frac{k_2}{m_1}u_1 + \frac{k_2}{m_1}u_3 - \frac{c}{m_1}u_1$$

$$u_3' = y' = u_4$$

$$u_4' = y'' = \frac{k_2}{m_2}x - \frac{k_2}{m_2}y - \frac{k_3}{m_2}y - \frac{c}{m_2}y = \frac{k_2}{m_2}u_1 - \frac{k_2}{m_2}u_3 - \frac{k_3}{m_2}u_3 - \frac{c}{m_2}u_3$$

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ we see,

$$\vec{u}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \left(-\frac{k_1}{m_1} - \frac{k_2}{m_1} - \frac{c}{m_1}\right) & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & \left(-\frac{k_2}{m_2} - \frac{k_3}{m_2} - \frac{c}{m_2}\right) & 0 \end{pmatrix} \vec{u}$$

5.1 5a **3 / 6**

- **- 0 pts** Correct
- **- 1 pts** Minor errors with mass
- **- 2 pts** Errors with signs
- **- 2 pts** Had damping terms for other masses
- **- 4 pts** No attempt at damping terms
- ✓ **- 3 pts Had damping attached to position instead of velocity**
- **- 4 pts** Had $$c$$ present on all terms?
- **- 4 pts** Had damping listed as a constant, tried to write as inhomogeneous?
- **- 5 pts** Serious mistakes
- **- 2 pts** One damping term incorrect
- **- 2 pts** Attached $$x_1,x_2$$ to $$c$$ in the matrix (but in the correct places)?
- **- 2 pts** Missing one $$\frac{k_2}{m_i}$$ term
- **- 2 pts** Simple mistake with $$x'$$ vs $$y'$$
- **- 3 pts** No mass terms
- **- 2 pts** Included $$x,x',y,y'$$ terms in the matrix
- **- 2 pts** Errors with $$k_i$$'s

(b) [4 pts] Think about the actual motion of such a damped system. If c is very small (think 'underdamped'), use your physical intuition to deduce the types of eigenvalues A must have (regardless of the exact values of the constants). Be as specific as you can be. Real or complex? If real, positive/negative/zero/unsure? If complex $\alpha + i\beta$, positive/negative/zero/unsure for both α and β?

- the eigenvalues will both be complex
- since it is underdamped, α is negative, and the eigenvalues will be complex conjugates of eachother, so β is positive for λ_1 and negative for λ_2. α is negative because $e^{\alpha t} \to 0$ for $t \to \infty$.

(c) [4 pts] Consider the same questions for the case when c is very large (think 'overdamped'), again using your physical intuition as far as possible.

- both eigenvalues will be real.
- both those eigenvalues will be negative, since $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$... must $\to 0$ for $t \to \infty$

5.2 5b **4 / 4**

✓ **- 0 pts** Correct

- 1 pts Believed in a fixed sign for $$\beta$$

- 2 pts Complex, unsure of $$\alpha$$ sign

- 2 pts Claimed $$\alpha>0$$

- 2 pts Claimed $$\alpha=0$$

- 4 pts Nothing helpful

- 2 pts "Complex and negative"

- 1 pts Claimed $$\alpha>0$$ to maintain the minus sign in the quadratic formula

- 3 pts "Small distinct eigenvalues, $$\alpha>0,\beta<0$$"

- 1 pts Claimed $$\alpha\leq 0$$ only

(b) [4 pts] Think about the actual motion of such a damped system. If c is very small (think 'underdamped'), use your physical intuition to deduce the types of eigenvalues A must have (regardless of the exact values of the constants). Be as specific as you can be. Real or complex? If real, positive/negative/zero/unsure? If complex $\alpha + i\beta$, positive/negative/zero/unsure for both α and β?

- the eigenvalues will both be complex
- since it is underdamped, α is negative, and the eigenvalues will be complex conjugates of eachother, so β is positive for λ_1 and negative for λ_2. α is negative because $e^{\alpha t} \to 0$ for $t \to \infty$.

(c) [4 pts] Consider the same questions for the case when c is very large (think 'overdamped'), again using your physical intuition as far as possible.

- both eigenvalues will be real.
- both those eigenvalues will be negative, since $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$... must $\to 0$ for $t \to \infty$

5.3 5c **3 / 4**

- **- 0 pts** Correct
- ✓ **- 1 pts Implied only two eigenvalues**
- **- 2 pts** Real, but sign wrong (or unmentioned/unclear)
- **- 4 pts** Nothing helpful
- **- 3 pts** Believed complex with negative $$\beta$$
- **- 2 pts** Believed half of the eigenvalues were complex

To consider a case where c is 'moderate', suppose that the constants $m_1, m_2, k_1, k_2, k_3, c$ have been chosen so that the resulting linear system is as follows:

$$\vec{\mathbf{u}}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -4 & -2 \end{pmatrix} \vec{\mathbf{u}}$$

(d) [4 pts] Verify that $\lambda = -1$ is an eigenvalue with associated eigenvector $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

This is only true if $(A-\lambda I)\vec{v} = \vec{0}$

Checking: $(A-\lambda I)\vec{v} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 0 & -4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ -2+1+1 \\ 1-1 \\ 3-4+1 \end{pmatrix} = \vec{0}$

$\therefore \lambda = -1$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is the associated eigenvector.

(e) [6 pts] In fact, $\lambda = -1$ is a *repeated* root. Verify that $\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector for $\lambda = -1$ corresponding to $(A - \lambda I)^2$.

Need $(A-\lambda I)^2\vec{w} = \vec{0}$:

Checking: • $(A-\lambda I)^2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}$ (by computer)

• $(A-\lambda I)^2\vec{w} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \vec{0}$

$\Rightarrow \lambda = -1$ has generalized eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

5.4 5d 4 / 4

✓ - 0 pts Correct

To consider a case where c is 'moderate', suppose that the constants $m_1, m_2, k_1, k_2, k_3, c$ have been chosen so that the resulting linear system is as follows:

$$\vec{\mathbf{u}}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -4 & -2 \end{pmatrix} \vec{\mathbf{u}}$$

(d) [4 pts] Verify that $\lambda = -1$ is an eigenvalue with associated eigenvector $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

This is only true if $(A-\lambda I)\vec{v} = \vec{0}$

Checking: $(A-\lambda I)\vec{v} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 0 & -4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ -2+1+1 \\ 1-1 \\ 3-4+1 \end{pmatrix} = \vec{0}$

$\therefore$ $\lambda=-1$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is the associated eigenvector.

(e) [6 pts] In fact, $\lambda = -1$ is a *repeated* root. Verify that $\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector for $\lambda = -1$ corresponding to $(A - \lambda I)^2$.

Need $(A-\lambda I)^2\vec{w} = \vec{0}$:

Checking: • $(A-\lambda I)^2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}$ (by computer)

• $(A-\lambda I)^2\vec{w} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \vec{0}$

$\Rightarrow$ $\lambda=-1$ has generalized eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

5.5 5e 6 / 6

✓ **- 0 pts** Correct

- 1 pts Incorrect value of $$(A - \lambda I)^2$$.

(f) [6 pts] Verify that $\lambda = -1+2i$ is a complex eigenvalue with associated complex eigenvector $\vec{\mathbf{z}} = \begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$.

Need $(A-\lambda I)\vec{z} = (A+(1-2i)I)\vec{z} = 0$

$$(A-\lambda I)\vec{z} = \begin{pmatrix} 1-2i & 1 & 0 & 0 \\ -2 & -1-2i & 1 & 0 \\ 0 & 0 & 1-2i & 1 \\ 3 & 0 & -4 & -1-2i \end{pmatrix}\begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} (1-2i)(1+2i)-5 \\ -2(1+2i)-5(-1-2i)-3-6i \\ (-3-6i)(1-2i)+15 \\ 3(1+2i)-4(-3-6i)+15(-1-2i) \end{pmatrix}$$

$$= \begin{pmatrix} 5-5 \\ -2-4i+5+10i-3-6i \\ -3-6i+6i-12+15 \\ 3+6i+12+24i-15-30i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$$

So $\vec{z}$ is a complex eigenvector for $\lambda = -1+2i$

(g) [2 pts] What is the fourth (and final) eigenvalue for A?

$\lambda = -1-2i$ (complex conjugate)

5.6 5f 6 / 6

✓ **- 0 pts** Correct

- 3 pts Checked eigenvalue but not eigenvector

- 3 pts Computed A- lambda I but did not apply to the vector

- 6 pts Incorrect or no work

(f) [6 pts] Verify that $\lambda = -1+2i$ is a complex eigenvalue with associated complex eigenvector $\vec{\mathbf{z}} = \begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$.

Need $(A-\lambda I)\vec{z} = (A+(1-2i)I)\vec{z} = 0$

$$(A-\lambda I)\vec{z} = \begin{pmatrix} 1-2i & 1 & 0 & 0 \\ -2 & -1-2i & 1 & 0 \\ 0 & 0 & 1-2i & 1 \\ 3 & 0 & -4 & -1-2i \end{pmatrix}\begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} (1-2i)(1+2i) - 5 \\ -2(1+2i) - 5(-1-2i) - 3-6i \\ (-3-6i)(1-2i) + 15 \\ 3(1+2i) - 4(-3-6i) + 15(-1-2i) \end{pmatrix}$$

$$= \begin{pmatrix} 5-5 \\ -2-4i+5+10i-3-6i \\ -3-6i+6i-12+15 \\ 3+6i+12+24i-15-30i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$$

So $\vec{z}$ is a complex eigenvector for $\lambda = -1+2i$

(g) [2 pts] What is the fourth (and final) eigenvalue for A?

$\lambda = -1-2i$ (complex conjugate)

5.7 5g **2 / 2**

- ✓ **- 0 pts** Correct
- **- 2 pts** Incorrect / no answer
- **- 1 pts** Click here to replace this description.

(h) [10 pts] Write down the general solution for the system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$. Here are all of the eigenvalue/eigenvector pairs you have been given:

$$\lambda = -1 \implies \vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{\mathbf{w}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ (for } (A-\lambda I)^2)$$

$$\lambda = -1+2i \implies \vec{\mathbf{z}} = \begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$$

we know the solution for $u' = Au$ when

$\lambda = -1$: is a repeated root

u_1: since $(A-\lambda I)\vec{v} = 0$, $u_1 = e^{tA} v = \boxed{e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}$ is a soln

u_2: since $(A-\lambda I)^2 \vec{w} = 0$, $u_2 = e^{tA} v$

$= e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t(A-\lambda I)\vec{w}\right] = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 0 & -4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right]$

$= e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right] = \boxed{e^{-t}\begin{pmatrix} 1+t \\ -t \\ 1+t \\ -t \end{pmatrix}}$

for $\lambda = -1+2i$:

we have $\vec{z} = \begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$ and $z(t) = e^{(\alpha + i\beta t)}\vec{z}$

(There is more space on the next page)

(More space for computations)

$$= e^{\alpha t}[\cos\beta t + i\sin\beta t]\vec{z} \quad (\text{Euler's formula})$$

$$= e^{-t}[\cos 2t + i\sin 2t]\begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \cos 2t + 2i\cos 2t + i\sin 2t - 2\sin 2t \\ -5\cos 2t - 5i\sin 2t \\ -3\cos 2t - 6i\cos 2t - 3i\sin 2t + 6\sin 2t \\ 15\cos 2t + 15i\sin 2t \end{pmatrix}$$

$$= e^{-t}\left[\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \\ -3\cos 2t + 6\sin 2t \\ 15\cos 2t \end{pmatrix} + i\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \\ -6\cos 2t - 3\sin 2t \\ 15\sin 2t \end{pmatrix}\right]$$

so $u_3 = re[\vec{z}(t)]$

$$= e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \\ -3\cos 2t + 6\sin 2t \\ 15\cos 2t \end{pmatrix}$$

$u_4 = Im[\vec{z}(t)]$

$$= e^{-t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \\ -6\cos 2t - 3\sin 2t \\ 15\sin 2t \end{pmatrix}$$

So, general solution

$u(t) = C_1 u_1 + C_2 u_2 + C_3 u_3 + C_4 u_4$

$$= C_1 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1+t \\ -t \\ 1+t \\ -t \end{pmatrix}$$

$$+ C_3 e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \\ -3\cos 2t + 6\sin 2t \\ 15\cos 2t \end{pmatrix}$$

$$+ C_4 e^{-t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \\ -6\cos 2t - 3\sin 2t \\ 15\sin 2t \end{pmatrix}$$

(i) [4 pts] Convert your solution for $\vec{\mathbf{u}}$ into a set of solutions for $x(t)$ and $y(t)$, and describe the long-term behavior of such solutions in the xy-plane.

$\Rightarrow u_1 = x(t),\ u_3 = y(t)$ so:

$$x(t) = e^{-t}[C_1 + C_2(1+t) + C_3(\cos 2t - 2\sin 2t) + C_4(2\cos 2t + \sin 2t)]$$

$$y(t) = e^{-t}[C_1 + C_2(1+t) + C_3(-3\cos 2t + 6\sin 2t) + C_4(-6\cos 2t - 3\sin 2t)]$$

for $x(t), y(t)$ as $t \to \infty$, $e^{-t} \to 0$ so

$x(t) \to 0,\ y(t) \to 0$

5.8 5h **10 / 10**

✓ **- 0 pts** **Correct**

- 2 pts error in solution associated to generalized eigenvector w

- 3 pts incorrect solution associated to w

- 5 pts Does not give real valued solutions associated to complex conjugate pair of eigenvectors

- 1 pts incorrect solution associated to v

- 2 pts error in solutions associated to complex conjugate pair

- 4 pts incorrect/incomplete solutions associated to complex conjugate pair

- 10 pts Incorrect / no work

(More space for computations)

$$= e^{\alpha t}[\cos\beta t + i\sin\beta t]\vec{z} \quad \text{(Euler's formula)}$$

$$= e^{-t}[\cos 2t + i\sin 2t]\begin{pmatrix} 1+2i \\ -5 \\ -3-6i \\ 15 \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \cos 2t + 2i\cos 2t + i\sin 2t - 2\sin 2t \\ -5\cos 2t - 5i\sin 2t \\ -3\cos 2t - 6i\cos 2t - 3i\sin 2t + 6\sin 2t \\ 15\cos 2t + 15i\sin 2t \end{pmatrix}$$

$$= e^{-t}\left[\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \\ -3\cos 2t + 6\sin 2t \\ 15\cos 2t \end{pmatrix} + i\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \\ -6\cos 2t - 3\sin 2t \\ 15\sin 2t \end{pmatrix}\right]$$

so $u_3 = re[\vec{z}(t)]$

$$= e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \\ -3\cos 2t + 6\sin 2t \\ 15\cos 2t \end{pmatrix}$$

$u_4 = Im[\vec{z}(t)]$

$$= e^{-t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \\ -6\cos 2t - 3\sin 2t \\ 15\sin 2t \end{pmatrix}$$

So, general solution

$u(t) = C_1u_1 + C_2u_2 + C_3u_3 + C_4u_4$

$$= C_1e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + C_2e^{-t}\begin{pmatrix} 1+t \\ -t \\ 1+t \\ -t \end{pmatrix} + C_3e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \\ -3\cos 2t + 6\sin 2t \\ 15\cos 2t \end{pmatrix} + C_4e^{-t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \\ -6\cos 2t - 3\sin 2t \\ 15\sin 2t \end{pmatrix}$$

(i) [4 pts] Convert your solution for $\vec{\mathbf{u}}$ into a set of solutions for $x(t)$ and $y(t)$, and describe the long-term behavior of such solutions in the xy-plane.

$\Rightarrow u_1 = x(t),\ u_3 = y(t)$ so:

$$x(t) = e^{-t}[C_1 + C_2(1+t) + C_3(\cos 2t - 2\sin 2t) + C_4(2\cos 2t + \sin 2t)]$$

$$y(t) = e^{-t}[C_1 + C_2(1+t) + C_3(-3\cos 2t + 6\sin 2t) + C_4(-6\cos 2t - 3\sin 2t)]$$

for $x(t), y(t)$ as $t \to \infty$, $e^{-t} \to 0$ so

$x(t) \to 0,\ y(t) \to 0$

5.9 5i 4 / 4

✓ - **0 pts** Correct

- **4 pts** Incorrect / no work

- **2 pts** Does not indicate long term behavior or incorrect assessment

- **2 pts** Incorrect solutions x(t), y(t)

- **1 pts** Does not indicate long term behavior approaches (0,0)

- **1 pts** error in one of the solutions

6. Consider the following non-linear system of 1st order differential equations:

$$x' = y - 2$$
$$y' = A\sin x - 2y + 4$$

where A is some arbitrary constant.

(a) [4 pts] Find the equilibrium points (there should be infinitely many of them).

step 1: solve
$$\begin{cases} x' = y - 2 = 0 \quad (1) = f(x,y) \\ y' = A\sin x - 2y + 4 = 0 \quad (2) = g(x,y) \end{cases}$$

(1) $y - 2 = 0 \Rightarrow y = 2$

(2) if $y = 2$, $A\sin x = 0$

note $A\sin x = 0$ for $x = \pi n$, for $n \in \mathbb{Z}$

so then equilibrium points have the form

$(\pi n, 2)$, for $n \in \mathbb{Z}$

6.1 6a **4 / 4**

- ✓ **+ 2 pts** y = 2
- ✓ **+ 2 pts** $$x = \pi k, k\in\mathbb Z$$
- **+ 1 pts** Only finitely many solutions for x
- **+ 0 pts** Incorrect or blank

(b) [8 pts] In the case that $A = 8$, use linearization to determine the type of each equilibrium point. Make sure your answer clearly covers all of the points.

$$J(x_0, y_0) = \begin{pmatrix} \delta f/\delta x & \delta f/\delta y \\ \delta g/\delta x & \delta g/\delta y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8\cos x & -2 \end{pmatrix}$$

at eq. points $(\pi n, 2)$, $J(2, \pi n) = \begin{pmatrix} 0 & 1 \\ 8\cos \pi n & -2 \end{pmatrix}$

Now we have $u' = \begin{pmatrix} 0 & 1 \\ 8\cos \pi n & -2 \end{pmatrix} u$ near $\vec{0}$

for n even, $8\cos \pi n = 8$, so $u' = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} u$

Note: Saddle when $D<0$, $D = -8 < 0$, so

eq. pts $(\pi n, 2)$ are <u>saddle points</u> when n is even.

for n odd, $8\cos \pi n = -1$, so $u' = \begin{pmatrix} 0 & 1 \\ -8 & -2 \end{pmatrix} u$

so $T = -2$, $D = 8$

$T^2 - 4D = 4 - 32 < 0$

so eq. pts $(\pi n, 2)$ are <u>spiral sinks</u> for n odd.

6.2 6b **8 / 8**

- ✓ **+ 4 pts** **Even multiples of $$\pi$$: saddle**
- ✓ **+ 4 pts** **Odd multiples of $$\pi$$: spiral sink**
- **+ 0 pts** Incorrect or blank
- **+ 4 pts** Med. partial credit (e.g. sign error/results backwards)
- **+ 2 pts** Partial credit
- **+ 1 pts** Compute Jacobian but no other progress

(c) [6 pts] There are precisely three values for A that can cause the linearization technique to be inconclusive at certain equilibrium points. Find these three values, and for each of them state which equilibrium points have inconclusive behavior.

inconclusive when $T^2-4D=0$, or when $D=0$

$T=2$, so $4-4D=0$

so $D=1$ gives inconclusive behavior.

Note $D=-A\cos\pi n=1$

① when $A=0$: $D=0$, $T>0$, so we have the equilibrium point $(0,2)$, which is a center, and is not generic, so this analysis is inconclusive for the original system.

② when $A=1$: $D=-\cos\pi n$, which means $D=1$ for n odd. equilibrium points are at $(\pi n, 2)$ (for n odd) have inconclusive behavior since it lies on the $T^2-4D=0$ curve.

③ when $A=-1$: $D=\cos\pi n$, which means $D=1$ for n even. equilibrium points in the form $(\pi n, 2)$, (for n even) have inconclusive behavior since it lies on the $T^2-4D=0$ curve.

So inconclusive behavior at $A=1,-1,0$

6.3 6c **6 / 6**

- ✓ **+ 1 pts** **$$A = 0$$**
- ✓ **+ 1 pts** **$$A = 1$$**
- ✓ **+ 1 pts** **$$A = -1$$**
- ✓ **+ 1 pts** **$$A = 0$$ is inconclusive for any $$k$$**
- ✓ **+ 1 pts** **$$A = 1$$ is inconclusive for odd $$k$$**
- ✓ **+ 1 pts** **$$A = -1$$ is inconclusive for even $$k$$**
- **+ 0 pts** Incorrect or blank
- **+ 2 pts** Some progress but not specific, or partial progress due to errors in (b)

(d) *Bonus: worth 5 points max* If A is some fixed value that is not one of the three 'bad' values you've found above, is it possible that *all* of the equilibrium points exhibit the *same* behavior? Explain.

This is not possible. This is because for the linearization, $D = -A\cos\pi n$. given $A \neq 0, -1$ or 1, D will oscillate between being $-A$ or A, depending on whether n is odd or even, since if n is even, $\cos\pi n = 1$ and if n is odd, $\cos\pi n = -1$. Since D oscillates between being negative and positive, the behavior will oscillate between a saddle point $(D<0)$, and other forms of behavior $(D>0)$. Therefore for any $A \neq 0, -1$ or 1, it is NOT possible that all equilibrium points exhibit the same behavior.

6.4 6d(bonus) 5 / 0

- ✓ **+ 5 pts** **Correct**
- **+ 3 pts** Right idea, but some issues with explanation
- **+ 2 pts** Believes non-saddles are always one type of sink
- **+ 2 pts** Believes non-saddles can be either source or sink
- **+ 1 pts** Believes non-saddles are always source
- **+ 1 pts** On the right track, but with some incomplete/incorrect elements
- **- 0 pts** Nothing helpful