1. (a) (20 points) Find two linearly independent solutions to $y'' - 6y' + 9y = 0$. Check that both of your solutions satisfy the equation.

$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2$

$\lambda = 3$

$y_1(t) = e^{3t}$ $\qquad$ $y_2(t) = te^{3t}$

$y_1'(t) = 3e^{3t}$ $\qquad$ $y_2'(t) = e^{3t} + 3te^{3t}$

$y_1''(t) = 9e^{3t}$ $\qquad$ $y_2''(t) = 3e^{3t} + 3e^{3t} + 9te^{3t}$

$= 6e^{3t} + 9te^{3t}$

$y'' - 6y' + 9y$ $\qquad$ $y'' - 6y' + 9y$

$\Rightarrow 9e^{3t} - 18e^{3t} + 9e^{3t} = 0$ $\qquad$ $\Rightarrow 6e^{3t} + 9te^{3t} - 6e^{3t} - 18te^{3t} + 9te^{3t} = 0$

(b) (5 points) Compute the Wronskian of the two solutions you found in part (a). Is it zero for any value of t?

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t} + 3te^{3t} \end{vmatrix}$$

$e^{3t}(e^{3t} + 3te^{3t}) - 3e^{3t}(te^{3t})$

$= e^{6t} + 3te^{6t} - 3te^{6t}$

$= e^{6t}$ $\Rightarrow$ never zero

$9-12+3$

2. (a) (5 points) Find the general solution to $y'' + 4y' + 3y = 0$.

$\lambda^2 + 4\lambda + 3 = 0$

$\lambda = \frac{-4 \pm \sqrt{16 - 4\times 3}}{2} = \frac{-4 \pm 2}{2}$ $\lambda = -1, -3$

$\therefore\ y_1(t) = e^{-t} \quad y_2(t) = e^{-3t}$

$y(t) = c_1 e^{-t} + c_2 e^{-3t}$

5/5

(b) (10 points) Write down a system of two first-order ODEs for unknown functions $y(t)$ and $v(t)$ which is equivalent to the ODE in part (a) (precisely: such that $y(t)$ solves the ODE in part (a) $\iff \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$ solves the first-order system).

Give your answer as two scalar equations, not as a matrix equation (you will put the answer in matrix form in part (c)).

$y' = v$

$y'' = -4y' - 3y = v'$

$v' = -4v - 3y$

10/10

$\therefore\ \begin{cases} y' = v \\ v' = -4v - 3y \end{cases}$

(c) (5 points) Write the system of two first-order ODEs from part (b) in matrix form

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = A \begin{bmatrix} y \\ v \end{bmatrix},$$

where A is a 2×2 matrix; give the matrix A explicitly.

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} ay + bv \\ cy + dv \end{bmatrix} = \begin{bmatrix} v \\ -4v - 3y \end{bmatrix}$$

$\therefore a = 0, b = 1, c = -3, d = -4$

$$\boxed{\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}}$$

5/5

(d) (5 points) The trace of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $a + d$, the sum of the entries along the "diagonal" of the matrix. Compute the trace and determinant of the matrix A from part (c).

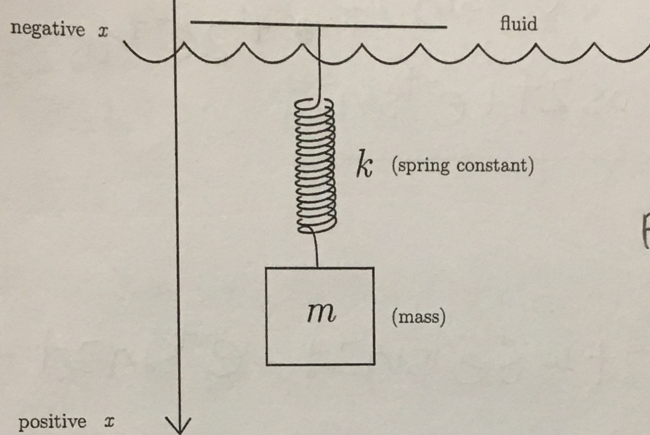
$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

5/5

$$\boxed{\begin{aligned} &\text{trace} = 0 + -4 = -4 \\ &\text{deterinant} = 0 \times -4 - (-3 \times 1) \\ &= 3 \end{aligned}}$$

3. In this problem, ignore units and treat all quantities as pure numbers. Also, ignore the force of gravity on the block.

Consider a block of mass m (constant) hanging from a spring. Let $x(t)$ be the position of the (center of mass of the) block at time t; orient the axis so that positive x points downward, and let $x = 0$ be the resting position of the spring with no block attached. The spring obeys Hooke's law, so the force of the spring on the block is $-kx(t)$, where k is a constant. The block and spring are in a fluid which provides a damping force $-\mu x'(t)$ on the block, where μ is another constant.



F=ma

(a) (10 points) Write down a harmonic motion equation which is satisfied by $x(t)$. *Make sure you check your answer for this part, because it will be used in the next parts of this question.*

10

Determine the damping constant c and natural frequency ω_0 of the equation. When is the equation underdamped, overdamped, and critically damped? Your answers should be in terms of the constants m, k, and μ.

$$-kx - \mu x' = mx''$$

$$mx'' + \mu x' + kx = 0$$

$$x'' + \frac{\mu}{m}x' + \frac{k}{m}x = 0$$

$$x'' + 2cx' + \omega_0^2 x = 0$$

↓ harmonic motion equation

$$2c = \frac{\mu}{m} \therefore \quad c = \frac{\mu}{2m} \text{ - damping constant}$$

$$\omega_0^2 = \frac{k}{m} \therefore \quad \omega_0 = \sqrt{\frac{k}{m}} \text{ natural frequency}$$

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0 \qquad \lambda = \frac{-2c \pm \sqrt{4c^2 - 4\omega_0^2}}{2}$$

$$\sqrt{4c^2 - 4\omega_0^2}$$

$\therefore$ critically damped : $c = \omega_0$
overdamped : $c > \omega_0$
underdamped : $c < \omega_0$

(b) (10 points) Set $m = 1$, $k = 5$, and $\mu = 2$. Assume that at $t = 0$, the position of the block is $x = -1$ and the velocity of the block is 3. Find $x(t)$ for all t.

10

$mx'' + \mu x' + kx = 0$

$x'' + 2x' + 5x = 0 \Rightarrow \lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4 - 4\times 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm i4}{2} = -1 \pm i2 = a \pm bi$

$x(t) = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt$

$a = -1 \quad b = 2$

$\therefore x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$x'(t) = -c_1 e^{-t}\cos 2t - 2c_1 e^{-t}\sin 2t - c_2 e^{-t}\sin 2t + 2c_2 e^{-t}\cos 2t$

$x(0) = c_1 = -1$

$x'(0) = -c_1 + 2c_2 = 3 \qquad 1 + 2c_2 = 3,\ c_2 = 1$

$\boxed{\therefore x(t) = -e^{-t}\cos 2t + e^{-t}\sin 2t}$

(c) (5 points) Find constants ~~A, λ, ω, and ϕ, with $A > 0$ and ϕ in the interval~~ $(-\pi, \pi]$, such that your solution to part (b) is equal to $Ae^{\lambda t}\cos(\omega t - \phi)$. Sketch the graph of your solution.

~~10~~ 4

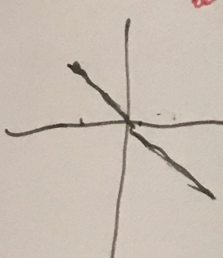
-45	-30	0	30	45	
$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	sin
$\frac{\sqrt{3}}{2}$			$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	cos

$x(t) = -e^{-t}(\cos 2t - \sin 2t)$

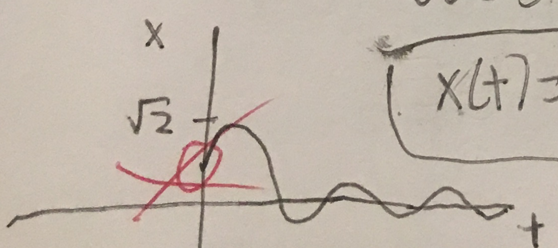
$A = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\tan\phi = \frac{1}{-1} = -1 \quad \phi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$\omega^2 = 5,\ \omega = \sqrt{5} \qquad \lambda = -1$



$\boxed{x(t) = \sqrt{2}e^{-t}\cos\left(\sqrt{5}t - \frac{3\pi}{4}\right)}$



shift by $\frac{3\pi}{4}$ to right

4. (a) (10 points) Find a particular solution to $y'' + 9y = \cos(t)$.

$y''+9y$

$y(t) = C_1\cos(t) + C_2\sin(t)$

$y'(t) = -C_1\sin t + C_2\cos t$

$\Rightarrow$

$y''(t) = -C_1\cos t - C_2\sin t$

$-C_1\cos t - C_2\sin t + 9C_1\cos t + 9C_2\sin t$

$= 8C_1\cos t + 8C_2\sin t = \cos(t) \quad \therefore C_1 = \frac{1}{8},\ C_2 = 0$

$y(t) = \frac{1}{8}\cos t$

+10

$-9\cos 3t - 9\sin 3t$

(b) (10 points) Find a particular solution to $y'' + 9y = 9\sec^2(3t)$. You may use the fact that $\int \sec(u)du = \ln|\sec(u) + \tan(u)|$ (ignoring constants of integration).

$y''+9y=0$

$y(t) = C_1\cos 3t + C_2\sin 3t$

$y_1 = \cos 3t,\ y_2 = \sin 3t$

$y_p(t) = v_1y_1 + v_2y_2 \qquad y_p(t) = v_1\cos 3t + v_2\sin 3t$

$y_p'(t) = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2' \qquad v_1'y_1 + v_2'y_2 = 0$

$= v_1y_1' + v_2y_2' \qquad v_1'$

$y_p''(t) = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$

$= -3v_1'\sin 3t - 9v_1\cos 3t + 3v_2'\cos 3t - 9v_2\sin 3t$

(more space on next page)

formula: $v_1' = \frac{-g(t)y_1}{W}$ $v_2' = \frac{g(t)y_2}{W}$

$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 3t + 3\sin^2 3t$

$= 3 \neq 0$

$v_1' = \frac{-\cancel{9}\sec^2 3t \cdot \cos 3t}{\cancel{3}} = -3\sec 3t$

$v_1 = -\ln|\sec(3t) + \tan(3t)|$

$v_2' = \frac{\cancel{9}\sec^2 3t \cdot \sin 3t}{\cancel{3}} = 3\tan 3t \sec 3t$ +8

$v_2 = \sec 3t$ $\boxed{y_p = -\ln|\sec 3t + \tan 3t|\cos 3t + \sec 3t \sin 3t}$ map

(c) (5 points) Find a particular solution to $y'' + 9y = 2\cos(t) - 3\sec^2(3t)$.

$= 2\left(\frac{1}{8}\cos t\right) - \frac{(y_p)}{3}$ +5

$y_p = \frac{1}{4}\cos t - \left(-\frac{1}{3}\ln|\sec 3t + \tan 3t|\cos 3t + \frac{1}{3}\sec 3t \sin 3t\right)$

$\boxed{= \frac{1}{4}\cos t + \frac{1}{3}\ln|\sec 3t + \tan 3t|\cos 3t - \frac{1}{3}\sec 3t \sin 3t}$