Math 33B
Winter 2017
Midterm Exam 2
2/27/2017
Time Limit: 50 Minutes

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Section: 2D

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Print your name legibly where requested on the top of this page, and print your initials on the top of every page, in case the pages become separated.

You should show your work clearly and concisely. If you need more space, use the back of the pages; clearly indicate when you have done this.

Draw a box around your final answer for each problem.

You may *not* use your books, notes, or any calculator on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	25	25
2	25	25
3	25	18
4	25	15
Total:	100	83

1. (a) (20 points) Find two linearly independent solutions to $y'' - 6y' + 9y = 0$. Check that both of your solutions satisfy the equation.

$$\lambda^2 - 6\lambda + 9 = 0$$
$$(\lambda - 3)^2 = 0$$
$$\lambda = 3$$

solutions: $\boxed{y_1 = e^{3t} \qquad y_2 = te^{3t}}$

check: $y_1'' - 6y_1' + 9y_1 = 0$

$$9e^{3t} - 6(3e^{3t}) + 9e^{3t} = 0$$
$$0 = 0 \checkmark$$

$y_2'' - 6y_2' + 9y_2 = 0$

$$9te^{3t} + 6e^{3t} - 6(3te^{3t} + e^{3t}) + 9(te^{3t}) = 0$$
$$9te^{3t} + 9te^{3t} - 18te^{3t} + 6e^{3t} - 6e^{3t} = 0$$
$$0 = 0 \checkmark$$

$$y_2' = 3te^{3t} + e^{3t}$$
$$y_2'' = 9te^{3t} + 3e^{3t} + 3e^{3t} = 9te^{3t} + 6e^{3t}$$

(b) (5 points) Compute the Wronskian of the two solutions you found in part (a). Is it zero for any value of t?

$$W(y_1, y_2) = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= y_1 y_2' - y_1' y_2$$
$$= e^{3t}(3te^{3t} + e^{3t}) - 3e^{3t}(te^{3t})$$
$$= 3te^{6t} + e^{6t} - 3te^{6t}$$

$\boxed{W = e^{6t}}$

This Function is not 0 for any value of t

2. (a) (5 points) Find the general solution to $y'' + 4y' + 3y = 0$.

$\lambda^2 + 4\lambda + 3 = 0$

$(\lambda + 3)(\lambda + 1) = 0$

$\boxed{y(t) = C_1 e^{3t} + C_2 e^{t}}$

5/5

(b) (10 points) Write down a system of two first-order ODEs for unknown functions $y(t)$ and $v(t)$ which is equivalent to the ODE in part (a) (precisely: such that $y(t)$ solves the ODE in part (a) $\iff$ $\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$ solves the first-order system).

Give your answer as two scalar equations, not as a matrix equation (you will put the answer in matrix form in part (c)).

$v = y'$

$y'' + 4y' + 3y = 0 \Rightarrow v' + 4v + 3y = 0$

$$\boxed{\begin{aligned} v' &= -4v - 3y \\ v &= y' \end{aligned}}$$

10/10

(c) (5 points) Write the system of two first-order ODEs from part (b) in matrix form

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = A \begin{bmatrix} y \\ v \end{bmatrix},$$

where A is a 2×2 matrix; give the matrix A explicitly.

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

5/5

(d) (5 points) The trace of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $a + d$, the sum of the entries along the "diagonal" of the matrix. Compute the trace and determinant of the matrix A from part (c).

$$\text{Trace}(A) = 0 - 4$$
$$= -4$$

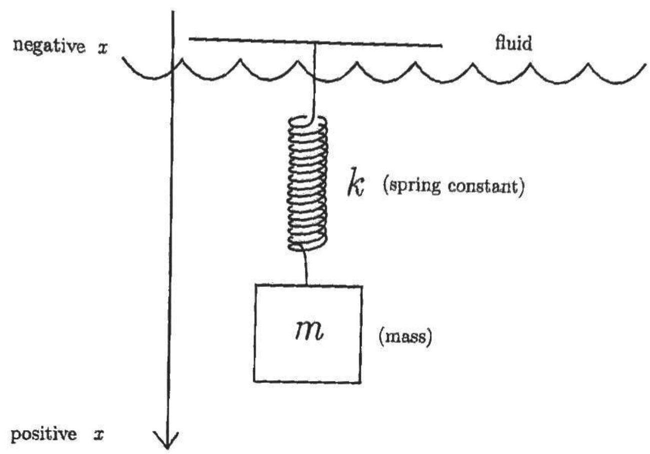
$$\det(A) = ad - bc$$
$$= 0(-4) - (1)(-3)$$
$$= 3$$

5/5

$$\boxed{\begin{aligned} \text{Trace}(A) &= -4 \\ \det(A) &= 3 \end{aligned}}$$

3. In this problem, ignore units and treat all quantities as pure numbers. Also, ignore the force of gravity on the block.

Consider a block of mass m (constant) hanging from a spring. Let $x(t)$ be the position of the (center of mass of the) block at time t; orient the axis so that positive x points downward, and let $x = 0$ be the resting position of the spring with no block attached. The spring obeys Hooke's law, so the force of the spring on the block is $-kx(t)$, where k is a constant. The block and spring are in a fluid which provides a damping force $-\mu x'(t)$ on the block, where μ is another constant.



(a) (10 points) Write down a harmonic motion equation which is satisfied by $x(t)$. *Make sure you check your answer for this part, because it will be used in the next parts of this question.*

5

Determine the damping constant c and natural frequency ω_0 of the equation. When is the equation underdamped, overdamped, and critically damped? Your answers should be in terms of the constants m, k, and μ.

F=ma

$$m x''(t) + \mu x'(t) + k x(t) = 0$$

$$x'' + \frac{\mu}{m} x' + \frac{k}{m} x$$

$$\boxed{\cancel{c = \sqrt{\frac{\mu}{m}}}}$$

$$w_0^2 = \frac{k}{m}$$

$$\boxed{w_0 = \sqrt{\frac{k}{m}}}$$

$$\lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = \frac{-\frac{\mu}{m} \pm \sqrt{(\frac{\mu}{m})^2 - 4\frac{k}{m}}}{2}$$

critically damped:

$$0 = \sqrt{(\frac{\mu}{m})^2 - 4(\frac{k}{m})}$$

$$\frac{\mu^2}{m^2} - 4\frac{k}{m} = 0$$

~~underdamped~~ :

$$\cancel{0 > \sqrt{(\frac{\mu}{m})^2 - 4(\frac{k}{m})}}$$

?

overdamped:

$$0 < \sqrt{\frac{\mu^2}{m^2} - 4\frac{k}{m}}$$

(b) (10 points) Set $m = 1$, $k = 5$, and $\mu = 2$. Assume that at $t = 0$, the position of the block is $x = -1$ and the velocity of the block is 3. Find $x(t)$ for all t.

9

$x'' + 2x' + 5x = 0$ $\quad$ $x(0) = -1$

$\lambda^2 + 2\lambda + 5 = 0$ $\quad$ $x'(0) = 3$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

$x(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

$x(0) = C_1 = -1$

$x'(t) = e^{-t}\cos(2t) + e^{-t}\sin 2t - C_2 e^{-t} \sin 2t + C_2 e^{-t}\cos(2t)$

$3 = 1 + C_2$

$C_2 = 2$

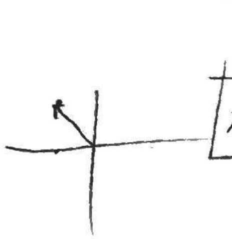
$x(t) = e^{-t}(-\cos(2t) + \cancel{2}\sin(2t))$

(c) (5 points) Find constants A, λ, ω, and ϕ, with $A > 0$ and ϕ in the interval $(-\pi, \pi]$, such that your solution to part (b) is equal to $Ae^{\lambda t}\cos(\omega t - \phi)$. Sketch the graph of your solution.

4

$A = \sqrt{C_1^2 + C_2^2}$ $\quad$ $\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right)$

$= \sqrt{2^2 + (-1)^2}$ $\quad$ $\boxed{\phi = \tan^{-1}(-2) + \pi}$



$\boxed{A = \sqrt{5}}$

$\boxed{\lambda = -1 \pm 2i}$

$w = \sqrt{\frac{5}{1}}$

$\boxed{w = \sqrt{5}}$

4. (a) (10 points) Find a particular solution to $y'' + 9y = \cos(t)$.

guess: $a\sin t + b\cos t = y(t)$

$y'(t) = a\cos t - b\sin t$

$y''(t) = -a\sin t - b\cos t$

$-a\sin t - b\cos t + 9(a\sin t + b\cos t) = \cos t$

$8a\sin t + 8b\cos t = \cos t$

$a = 0 \qquad b = \frac{1}{8}$

$\boxed{y_p = \frac{1}{8}\cos t}$

+10

(b) (10 points) Find a particular solution to $y'' + 9y = 9\sec^2(3t)$. You may use the fact that $\int \sec(u)du = \ln|\sec(u) + \tan(u)|$ (ignoring constants of integration).

$y_h'' + 9y_h = 0$

$\lambda^2 + 9 = 0$

$\lambda = \frac{\pm\sqrt{-36}}{2}$

$= \pm 3i$

+3

$y_h = C_1 e^{3i} + C_2 e^{-3i}$

$= \cos(3t) + i\sin(3t)$

$y_p = v_1\cos(3t) + v_2 i\sin(3t)$

$y' = v_1'\cos(3t) + v_2' i\sin(3t) - v_1 3\sin(3t) + v_2 3i\cos(3t)$

$0 = v_1'\cos(3t) + v_2' i\sin(3t)$

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$y_p' = -v_1 3\sin 3t + v_2 3i\cos 3t$

$y_p'' = -3v_1'\sin 3t - 9v_1\cos 3t + 3iv_2'\cos 3t - 9iv_2\sin 3t$

$y'' + 9y = 9\sec^2(3t)$

$-3v_1'\sin 3t - 9\cancel{v_1\cos 3t} + 3iv_2'\cos 3t - 9i\cancel{v_2\sin 3t} + \cancel{9v_1\cos 3t} + \cancel{9v_2\sin 3t} = 9\sec^2(3t)$

$-3v_1'\sin 3t + 3i\,v_2'\cos 3t = 9\sec^2(3t)$ $\quad 0 = v_1'\cos(3t) + v_2' i\sin 3t$

$v_1 = \sin 3t$ $\quad v_2 = \cos 3t$

$\boxed{y_p = \sin 3t(\cos 3t) + \cos(3t)\, i\sin(3t)}$

(c) (5 points) Find a particular solution to $y'' + 9y = 2\cos(t) - 3\sec^2(3t)$.

$\boxed{y_p \ \sin 3t\cos 3t + \cos 3t\, i\sin 3t + \cancel{\ }\cos t}$

wrong coefficient

just add our results

+2