1. Consider the ODE $y'(t) + y(t)^2 \sin(t) = 0$.

   (a) (20 points) Find the general solution to this ODE.

25

$y' = -y^2 \sin t$

$-y^{-2}$

$\frac{dy}{-y^2} = \sin t \, dt$

$y^{-1} = -\cos t + C$

$\frac{1}{y} = -\cos t + c$

$y = \frac{1}{-\cos t + c}$

$\boxed{y = \frac{-1}{\cos t + C}}$

$-y^2 \sin t = \frac{-\sin t}{(\cos t + c)^2}$

$-\sin t (\cos t + c)^{-2}$

$-\cos t (\cos t + c)^{-2} + 2(\cos t + c)^{-3} \sin t$

$\frac{-\cos t}{(\cos t + c)^2} + \frac{2 \sin t}{(\cos t + c)^3}$

$\frac{dy}{y^2} = -\sin t \, dt$

$-y^{-1} = \cos t + C$

✓

✓

   (b) (5 points) Find a solution satisfying $y(0) = 1$.

$y(0) = 1 = \frac{-1}{\cos(0) + C}$

$1 = \frac{-1}{1 + c}$

$1 + c = -1$

$c = -2$

$1 = \frac{1}{(-1) + C}$

$-1 + C = 1$

$c = 2$

$y(t) = \frac{-1}{\cos t + 2}$

✓

$\boxed{\therefore y(t) = \frac{-1}{\cos t - 2}}$

2. Consider the ODE $y'(t) + 2ty(t) = 2te^{-t^2}$.

(a) (18 points) Find the general solution to this ODE.

$$y' = -2ty + 2te^{-t^2}$$
$$u(t) = e^{\int 2t\,dt} = e^{t^2}$$
$$e^{t^2}y' = -2te^{t^2}y + 2te^{t^2}e^{-t^2}$$
$$u(t)y(t) = \int u(t)f(t)\,dt + C$$
$$e^{t^2}\cdot y(t) = \int e^{t^2}\cdot 2t\cdot e^{-t^2}\,dt + C$$
$$= \int 2t\,dt + C$$
$$= t^2 + C$$
$$\boxed{y(t) = e^{-t^2}t^2 + Ce^{-t^2}}$$

$$y' = -2te^{-t^2}t^2 + \cancel{2te^{-t^2}} - \cancel{2tCe^{-t^2}}$$
$$-2te^{-t^2}t^2 + \cancel{-2tCe^{-t^2}}$$

✓

(b) (2 points) Find a solution satisfying $y(0) = 0$.

$$y(0) = e^0\cdot 0^2 + C\cdot e^0 = 0$$
$$= 0 + C = 0$$
$$\therefore C = 0$$
$$\boxed{y(t) = e^{-t^2}t^2}$$

✓

(c) (5 points) Is the solution of the initial value problem in part 2 unique? Justify your answer.

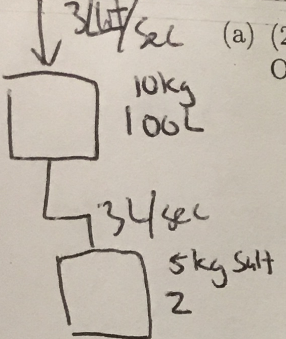
y ~~(t) = e^{-t^2}t^2~~

If we look at
F ~~φ~~ $= 2te^{-t^2} - 2ty$
~~and F'~~ both F and F'
~~and~~ are continuous in all region
therefore, if we set a rectangle with $(0,0)$ as the center,
with some width ~~and length~~,
~~a~~ it is unique

$$y'(t) = 2te^{-t^2} - 2ty(t)$$
$$F(y,t) = 2te^{-t^2} - 2ty \text{ — continous}$$
$$= \frac{2t}{e^{t^2}} - 2ty$$
$$F'(y,t) = \text{[illegible]} - 2t \text{ — contnous}$$

$y(t)$ is also continuous in all region as well as $y'(t)$ (stated before) solution ~~[illegible]~~

3. Tank 1 contains 100 liters of salt water, with 10 kilograms of salt at time $t = 0$. Tank 2 contains an unknown amount of salt water, with 5 kilograms of salt at time $t = 0$. Pure water is pumped into Tank 1 at a rate of 3 liters/sec. The water in Tank 1 passes through a pipe into Tank 2 at a rate of 3 liters/sec. No water leaves Tank 2; to make things simpler, assume the tanks have infinite capacity.



(a) (20 points) Translate the balance laws for Tank 1 and Tank 2 into a system of first-order ODEs for the salt content (in kilograms) $x(t)$ in Tank 1 and $y(t)$ in Tank 2.

Tank 1: rate in: $3 \times 0 = 0$ → no salt rate out: $-\frac{x(t)\,kg}{100\,L} \times 3\,L/sec = \frac{-3x(t)}{100}\,kg/sec$

✓ $\boxed{x'(t) = \frac{-3x(t)}{100}}$

Tank 2: rate in: rate out for tank 1 ∴ $\frac{3x(t)}{100}$ kg/sec

rate out: ~~y(t)~~ no water leave ∴ = 0

$\boxed{y'(t) = \frac{3x(t)}{100}}$

✓

20/20

(b) (5 points) Find explicit formulas for $x(t)$ and $y(t)$ as functions of time t.

$x'(t) = \frac{-3x(t)}{100}$ $x(t) = Ce^{\frac{-3}{100}t}$

$x(0) = 10 = Ce^{0}$

$\therefore C = 10$

✓ $\boxed{\therefore x(t) = 10e^{\frac{-3}{100}t}}$

$\frac{3}{10}e^{\frac{-3}{100}t} = \frac{3}{10\cancel{0}} \cdot 10e^{\frac{-3}{100}t}$

$y'(t) = \frac{3}{10\cancel{0}} \cdot \cancel{10}e^{\frac{-3}{100}t}$

5/5

$= \frac{3}{10}e^{\frac{-3}{100}t}$

$y(t) = \int \frac{3}{10}e^{\frac{-3}{100}t}\,dt$ $u = \frac{-3}{100}t$

$du = \frac{-3}{100}\,dt$

$= -10 \int e^{u}\,du$

$= -10e^{\frac{-3}{100}t} + C$ ✓

$\boxed{\therefore y(t) = -10e^{\frac{-3}{100}t} + 15}$

$y(0) = 5 = -10e^{0} + C$ $\therefore C = 15$

$= -10 + C = 5$

4. Consider the ODE $(3xy^2 - 2y) + (3x^2y - 2x)\frac{dy}{dx} = 0$.

(a) (20 points) Find the general solution to this ODE.

20

$P = 3xy^2 - 2y \quad Q = 3x^2y - 2x$

$\frac{\partial P}{\partial y} = 6xy - 2 \quad \frac{\partial Q}{\partial x} = 6xy - 2 \quad \therefore \text{Exact}$

$(3xy^2 - 2y)dx + (3x^2y - 2x)dy = 0$

$F = \int 3xy^2 - 2y \, dx = \frac{3x^2}{2}y^2 - 2yx + \phi(y)$

$P = 3x^2y - 2x + \phi'(y) = 3x^2y - 2x$

$\therefore \phi'(y) = 0$

$\phi(y) = 0$

$\boxed{F_{(x,y)} = \frac{3x^2}{2}y^2 - 2yx = C}$ ✓

5

(b) (5 points) Check that your solution satisfies the ODE.

$P = \frac{\partial F}{\partial x}, \; P = 3xy^2 - 2y \quad \frac{\partial F}{\partial x} = 3xy - 2y$ ✓

$Q = \frac{\partial F}{\partial y}, \; Q = 3x^2y - 2x, \; \frac{\partial F}{\partial y} = 3x^2y - 2x$ ✓

✓ Since $Q = \frac{\partial F}{\partial y}$, and $P = \frac{\partial F}{\partial x}$,
so solution satisfies the ODE