Math 33B
Winter 2017
Midterm Exam 1
2/1/2017
Time Limit: 50 Minutes

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Section: [illegible]

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Print your name legibly where requested on the top of this page, and print your initials on the top of every page, in case the pages become separated.

You should show your work clearly and concisely. If you need more space, use the back of the pages; clearly indicate when you have done this.

Draw a box around your final answer for each problem.

You may *not* use your books, notes, or any calculator on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	25	25
2	25	25
3	25	25
4	25	25
Total:	100	100

1. Consider the ODE $y'(t) + y(t)^2 \sin(t) = 0$.

   (a) (20 points) Find the general solution to this ODE.

$\Rightarrow y' + y^2 \sin(t) = 0.$

$\Rightarrow y' = -y^2 \sin(t). \Rightarrow$ Bernoulli equation: $y'(t) + a(t)y(t) = b(t)y(t)^n$.

(also separable)

$\Rightarrow z = y^{-1}$

$z' = -y^{-2}y'$

$\Rightarrow y^{-2}y' = -\sin(t).$

$\Rightarrow -z' = -\sin t$

$\Rightarrow z' = \sin t.$

$\Rightarrow z = \int \sin t \, dt$

$\Rightarrow -\cos t + C$

$\Rightarrow \frac{1}{y} = -\cos t + C$

$\boxed{y(t) = \frac{1}{-\cos t + C}}$

DC:

$y' = -y^2 \sin(t).$

$\int \frac{dy}{y^2} = \int -\sin t \, dt$

$\Rightarrow -\frac{1}{y} = \cos t + C$

$y = \frac{-1}{\cos t + C}.$

$\frac{+(-\sin t)}{(\cos t + c)^2} + \frac{\sin t}{(\cos t + c)^2} \Rightarrow 0 \checkmark. \Rightarrow$

$\Rightarrow \frac{-1}{(\cos t + c)^2}$

$\frac{1}{(-\cos t + c)}$

y^{-2}

$\Rightarrow -1 y^{-1}$

$\Rightarrow -\frac{1}{y}$

$\Rightarrow$

$\cos^2 t + \sin^2 t = 1$

$\Rightarrow 2\cos t(-\sin t)$

$+2\sin t(\cos t) = 0$

$\checkmark$.

DC:

$\frac{-(\sin t)}{(-\cos t + c)^2} + \frac{\sin t}{(-\cos t + c)^2} = 0 \checkmark.$

25

(b) (5 points) Find a solution satisfying $y(0) = 1$.

$1 = \frac{1}{-\cos(0) + C}.$

$\Rightarrow 1 = \frac{1}{-1 + C}.$

$\Rightarrow -1 + C = 1$

$\Rightarrow C = 2.$ ✓

$\Rightarrow \boxed{y(t) = \frac{1}{2 - \cos t}}$ $\Rightarrow$ DC: $\frac{(-)-(\sin t)}{(2-\cos t)^2} + \frac{\sin t}{(2-\cos t)^2} = 0$

2. Consider the ODE $y'(t) + 2ty(t) = 2te^{-t^2}$.

(a) (18 points) Find the general solution to this ODE.

$\Rightarrow \quad y' + 2ty(t) = 2te^{-t^2}$

Integrating factor $u(t) = e^{\int 2t\,dt} = e^{t^2}$

$\Rightarrow e^{t^2}(y' + 2ty(t)) = (e^{t^2}y)' = 2t(\cancel{e^{-t^2}})(\cancel{e^{t^2}})$

$\Rightarrow \int (e^{t^2}y)'\,dt = \int 2t\,dt$

DC: ✓

$\Rightarrow e^{t^2}y(t) = t^2 + C$

$\Rightarrow \boxed{y(t) = t^2e^{-t^2} + Ce^{-t^2}}$

$\Rightarrow 2te^{-t^2} - \cancel{2t^3e^{-t^2}} - 2tCe^{-t^2} + 2t(t^2e^{-t^2} + Ce^{-t^2})$ ✓

$2te^{-t^2} + t^2(-2t)e^{-t^2} - 2tCe^{-t^2}$ ✓ $-2t^3e^{-t^2}$ $+ 2t^3e^{-t^2} + 2tCe^{-t^2}$

(b) (2 points) Find a solution satisfying $y(0) = 0$.

$\Rightarrow y(0) = 0 = 0 + Ce^{0} \Rightarrow C = 0$

$\Rightarrow \boxed{y(t) = t^2e^{-t^2}}$

DC: $y(0) = 0$ ✓

$\Rightarrow 2te^{-t^2} + t^2(-2t)e^{-t^2}$ $-2t^3e^{-t^2} + 2t^3e^{-t^2}$ ✓

(c) (5 points) Is the solution of the initial value problem in part 2 unique? Justify your answer.

$\boxed{\text{Yes.}}$

$y'(t) = -2ty(t) + 2te^{-t^2}$

$\hookrightarrow y'(t) = a(t)y(t) + g(t)$, a linear first order ODE,

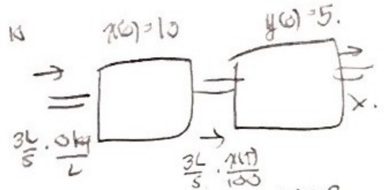
where $a(t) = -2t$, $g(t) = 2te^{-t^2}$.

$\Rightarrow$ Both $a(t)$ & $g(t)$ are continuous across the plane

so the hypotheses for the Existence & Uniqueness Theorem for linear ODE's are satisfied in any rectangle [illegible] the initial pt. $(0,0)$ & the theorem guarantees a unique [illegible] through this pt.

3. Tank 1 contains 100 liters of salt water, with 10 kilograms of salt at time $t = 0$. Tank 2 contains an unknown amount of salt water, with 5 kilograms of salt at time $t = 0$. Pure water is pumped into Tank 1 at a rate of 3 liters/sec. The water in Tank 1 passes through a pipe into Tank 2 at a rate of 3 liters/sec. No water leaves Tank 2; to make things simpler, assume the tanks have infinite capacity.

   (a) (20 points) Translate the balance laws for Tank 1 and Tank 2 into a system of first-order ODEs for the salt content (in kilograms) $x(t)$ in Tank 1 and $y(t)$ in Tank 2.



$V_x(0)=100L$ $V_y(0)=?$

$x(0)=10.$ $y(0)=5$

No change in volume since 3L in → 3L out.

Change in salt content = Rate in − Rate out

$$\frac{dx}{dt} = \cancel{\frac{3L}{s}\cdot\frac{0kg}{L}} - \frac{3L}{s}\cdot\frac{x(t)}{100}\frac{kg}{L} \Rightarrow \boxed{\begin{aligned}\frac{dx}{dt} &= -\frac{3x(t)}{100}\\ \frac{dy}{dt} &= \frac{3x(t)}{100}\end{aligned}} \quad x(0)=10,\ y(0)=5$$

$$\frac{dy}{dt} = \frac{3L}{s}\cdot\frac{x(t)\,kg}{100\,L} \cancel{- \frac{0L}{s}\frac{y(t)}{V_y}}.$$

20/20

   (b) (5 points) Find explicit formulas for $x(t)$ and $y(t)$ as functions of time t.

① $\frac{dx}{dt} = \frac{-3x(t)}{100}$

$\Rightarrow \int \frac{dx}{x} = \int -\frac{3}{100}\,dt$

$\Rightarrow \ln|x| = -\frac{3}{100}t + C_1$

$\Rightarrow x(t) = Ce^{-\frac{3}{100}t}$

$\Rightarrow x(0) = 10 = C$ ✓

$\Rightarrow C = 10.$

$\Rightarrow \boxed{x(t) = 10e^{-\frac{3}{100}t}}$

$\Rightarrow$ check: $x'(t) = -\frac{3}{10}e^{-\frac{3}{100}t}$

$= -\frac{3}{100}$ ✓

$x(0) = 10e^0 \Rightarrow 10$ ✓

② $\frac{dy}{dt} = \frac{3x(t)}{100}$

$\Rightarrow \frac{dy}{dt} = \frac{3(10)}{100}e^{-\frac{3}{100}t}$

$\frac{3}{10}e^{-\frac{3}{100}t}$

$\Rightarrow \int \frac{dy}{dt} = \int \frac{3}{10}e^{-\frac{3}{100}t}\,dt$

$\Rightarrow y(t) = \frac{3}{10}\left(-\frac{100}{3}\right)e^{-\frac{3}{100}t} + C_2$

$\Rightarrow -10e^{-\frac{3}{100}t} + C_2.$

$\Rightarrow y(0) = 5 = -10e^0 + C_2$

$\Rightarrow 5 = -10 + C_2$

$\Rightarrow C_2 = 15$

$\Rightarrow \boxed{y(t) = -10e^{-\frac{3}{100}t} + 15}$

5/5

DE: $y' = -10\left(-\frac{3}{100}e^{-\frac{3}{100}t}\right)$

$\left(\frac{3}{10}\right)\frac{3}{100}\left(10e^{-\frac{3}{100}t}\right)$

✓ ✓

$y(0) = -10 + 15 = 5$ ✓

4. Consider the ODE $(3xy^2 - 2y) + (3x^2y - 2x)\frac{dy}{dx} = 0$. $\Rightarrow (3xy^2-2y)dx + (3x^2y-2x)dy = 0$

(a) (20 points) Find the general solution to this ODE.

20

$\Rightarrow$ Check: $\Rightarrow$

$P = 3xy^2 - 2y \quad \frac{\partial P}{\partial y} = 6xy - 2$

$Q = 3x^2y - 2x \quad \frac{\partial Q}{\partial x} = 6xy - 2$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so the differential form is exact.

$\frac{\partial F}{\partial x} = 3xy^2 - 2y$

$\Rightarrow F(x,y) = \int(3xy^2 - 2y)dx$

$\Rightarrow \frac{3}{2}x^2y^2 - 2xy + \phi(y)$

$\Rightarrow \frac{\partial F}{\partial y} = Q = 3x^2y \cancel{- 2x} + \phi'(y) = 3x^2y \cancel{- 2x}$

$\Rightarrow \phi'(y) = 0$

$\Rightarrow \phi(y) = \int 0\, dy = 0$

$F(x,y) = \boxed{\frac{3}{2}x^2y^2 - 2xy = C}$

(b) (5 points) Check that your solution satisfies the ODE. 5

$\Rightarrow \frac{d}{dx}F(x,y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = (3xy^2 - 2y) + (3x^2y - 2x)\frac{dy}{dx} = \frac{d}{dx}(C) = 0$ ✓

Solution satisfies ODE. ✓