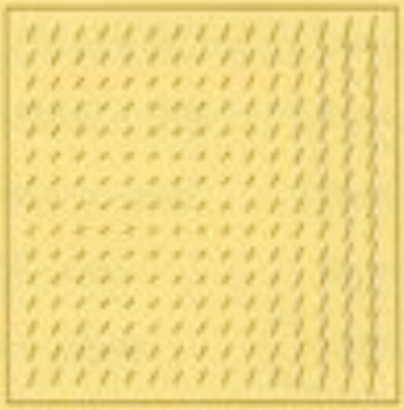
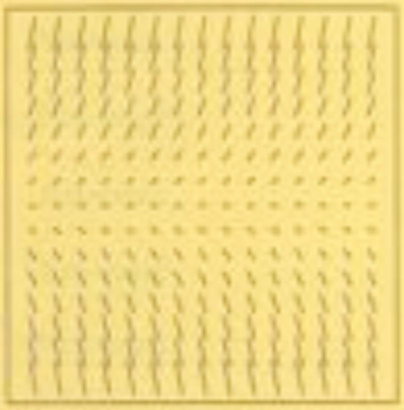
Question 1. (10 pts) Please match the following slope fields with their associated differential equations. Circle the correct answers.

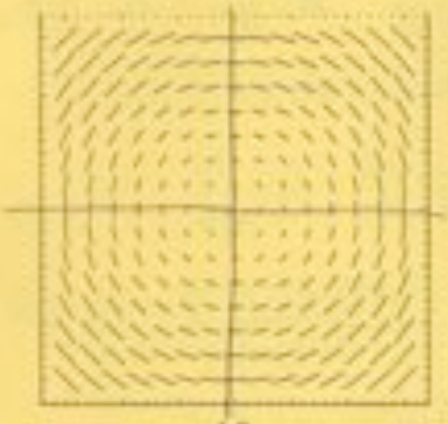
(a)



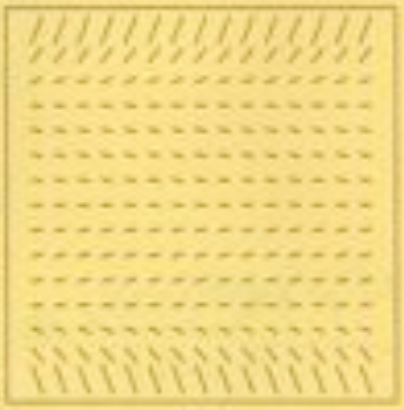
(b)



(c)



(d)



(e)

1) $y' = \frac{-x}{y}$ a b c d e

2) $y' = y(y-3)(y+3)$ a b c d e

3) $y' = y^2 + x^2 + e^x$ a b c d e

4) $y' = \sin x + \cos y - 2$ a b c d e

5) $y' = y$ a b c d e

10

Question 2. (10 pts) Which of the following differential 1-forms ω are exact on the domain $\mathbb{R}^2 \setminus (0,0)$? Circle the correct answer.

a) $\omega := dF$, for $F(x,y) = 2x^2 + y^2$	Exact	Not-exact
b) $\omega := dF$, for $F(x,y) = \frac{x-y}{x^2+y^2}$	Exact	Not-exact
c) $\omega := (x+y)dx + (x-y)dy$	Exact	Not-exact
d) $\omega := x dx + (y-x)dy$	Exact	Not-exact
e) $\omega := \frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy$	Exact	Not-exact

a) $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$

$= 4x\,dx + 2y\,dy$

$\frac{\partial P}{\partial y} = 0 \quad \frac{\partial Q}{\partial x} = 0$

b) $dF = \frac{(x-y)2x - (x^2+y^2)1}{(x^2+y^2)^2}dx + \frac{(x-y)2y + (x^2+y^2)1}{(x^2+y^2)^2}dy$

$= \frac{2x^2 - 2xy - x^2 - y^2}{(x^2+y^2)^2}dx + \frac{x^2+y^2+2xy-2y^2}{(x^2+y^2)^2}dy$

$\frac{\partial P}{\partial y} = (2x^2 - 2xy - x^2 - y^2)$

c) $\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 1$

d) $\frac{\partial P}{\partial y} = 0 \quad \frac{\partial Q}{\partial x} = -1$

e) $\frac{\partial P}{\partial y} = \frac{y(2y) - (x^2+y^2)1}{(x^2+y^2)^2} = \frac{2y^2 - x^2 - y^2}{} = y^2 - x^2$

$\frac{\partial Q}{\partial x} = \frac{-x(2x) - (x^2+y^2)\cdot -1}{(x^2+y^2)^2} = \frac{-2x^2 + x^2 + y^2}{} = y^2 - x^2$

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Question 3. Consider the initial value problem

$$y' = \sin y, \quad y(0) = 3.$$

Remember: when using any theorem from class, make sure to explain why it applies.

a) (2 pts) Does there exist a solution $y(t)$ to this initial value problem? Why?

There exists a solution to this initial value problem by the Existence theorem, which states that if any function $f(t,y)$ is continuous on a rectangle R, then that function has a solution. $\sin y$ is continuous on all of $\mathbb{R}^2$, and therefore there exists a solution.

+2

$\cancel{\frac{dy}{dt} = \sin y}$ $\cancel{\frac{dy}{\sin y} = dt}$

$\cancel{y = e^{\int \sin y \, dt} = Ae^{\sin y}}$ $\cancel{\int dy \csc y = \int dt}$

$\cancel{t = \underline{\qquad}}$

b) (2 pts) If so, how many solutions exist? Why?

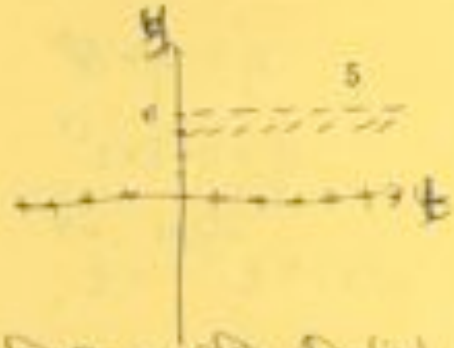
$\frac{\partial f}{\partial y} = \cos y$ which is continuous on all $\mathbb{R}^2$.

Therefore by the uniqueness theorem there exists a unique solution to the initial value problem, since the function $f(t,y)$ is continuous & it's derivative $\frac{\partial f}{\partial y}$ is also continuous.

+2

4

$\sin 3 \quad \sin \pi$



c) (4 pts) If $y(t)$ is a solution to the initial value problem,

$$y' = \sin y, \quad y(0) = 3,$$

what is $\lim_{t\to\infty} y(t)$? Justify your response.

~~The limit as $t \to \infty$ of $y(t)$ will simply be π, since the function will vary reasonably (the function's slope) to sin it will remain around 3.~~

The limit as $t \to \infty$ of $y(t)$ will be π, because when $y = 3$, then $\sin y = \sin 3$ which is slightly positive. This will approach $\sin y$ towards $\sin \pi$, which is 0. There is a horizontal asymptote at $\sin \pi$ because it is equal to 0, meaning the slope is equal to 0 and therefore $y(t)$ will not move away from this value.

+4

d) (4 pts) If $y(t)$ is a solution to the above initial value problem, then for which value(s) of t does $y(t) = -1$? Justify your answer.

~~$y(t) = -1$ will never occur because the slope will never be negative for long enough for the function to reach -1.~~

$y(t) = -1$ will never occur, because the function will start at $y(0) = 3$, it will increase slightly until $y(t) = \pi$, and then remain there to infinity. There will never be a time where $y'(t)$ is negative, and therefore no way for $y(t)$ to approach -1 from 3.

+4

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Question 4. (10 pts) Solve the initial value problem:

$$y' = y + 2xe^{2x}, \quad y(0) = 3.$$

$\frac{dy}{dx} = y + 2xe^{2x}$

var of param: $y_h' = y_h \quad y_h = e^{\int 1 dx} = e^x$

$v' = \frac{2xe^{2x}}{e^x} = 2xe^x$

$v = e^x$

u	dv
2x	e^x
2	e^x
du	v

$V = \int 2xe^x = 2xe^x - \int 2e^x = 2xe^x - 2e^x + C$

$y(x) = e^x(2xe^x - 2e^x + C)$

$y(x) = 2xe^{2x} - 2e^{2x} + Ce^x$

$y(0) = 3 = 0 - 2 + C$

$C = 5$

$\boxed{y(x) = 2xe^{2x} - 2e^{2x} + 5e^x}$

checked w/ integration factor on other page

10

Question 5. (12 pts) Solve the following initial value problem for an equation of the form $F(x, y) = C$ that defines y implicitly as a function of x:

$$y' = \frac{(2x\cos y + 3x^2 y)}{(y + x^2 \sin y - x^3)}, \quad y(0) = 2.$$

$$\frac{dy}{dx} = \frac{2x\cos y + 3x^2 y}{y + x^2 \sin y - x^3}$$

$$(y + x^2 \sin y - x^3)\,dy = (2x\cos y + 3x^2 y)\,dx$$

$$(2x\cos y + 3x^2 y)\,dx - (y + x^2 \sin y - x^3)\,dy = 0$$

$$\frac{\partial P}{\partial y} = -2x\sin y + 3x^2 \qquad \frac{\partial Q}{\partial x} = -2x\sin y + 3x^2$$

exact ✓

$$\frac{\partial F}{\partial x} = 2x\cos y + 3x^2 y$$

$$F = x^2 \cos y + x^3 y + \varphi(y)$$

$$\frac{\partial F}{\partial y} = -x^2 \sin y + x^3 + \varphi'(y) = -y - x^2 \sin y + x^3$$

$$\varphi'(y) = -y$$

$$\varphi(y) = \frac{-y^2}{2}$$

$$F = x^2 \cos y + x^3 y - \frac{y^2}{2} = C$$

when $x=0$, $y(x) = \cancel{3}\,2$

$$0\cos 3 + 0 - \frac{9}{2} = C$$

$$C = -\frac{9}{2}$$

$$\boxed{F(x,y) = x^2 \cos y + x^3 y - \frac{y^2}{2} = -\frac{9}{2}}$$

+12

Question 6. (6 pts) The differential form

$$\omega := 2y dx + (x + y) dy$$

is not exact. Find an integration factor of the form $\mu(x, y) = \mu(y)$ so that the product $\mu(y) \cdot \omega$ is equal to dF for some continuously differentiable function $F(x, y)$ on the domain $(0, \infty) \times (0, \infty)$. Note: you **do not** need to find this function F, just find $\mu(y)$.

$\frac{\partial P}{\partial y} = 2 \quad \frac{\partial Q}{\partial x} = 1$

not exact,

$\mu(y) = e^{-\int g(x) dx}$

$g(x) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= \frac{1}{2y}(2 - 1)$

$g(x) = \frac{1}{2y}$

$\mu(y) = e^{-\int \frac{1}{2y}} = e^{-\frac{1}{2}\ln y} = y^{-1/2}$

$\boxed{\mu(y) = y^{-1/2}}$

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