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1. (20 points) Prove that an overdumped solution of $my'' + \mu y' + ky = 0$ can cross the time axis no more than once, regardless of the initial conditions.

An overdamped solution of $my'' + \mu y' + ky = 0$ is also an overdamped solution of the equation modified to

$$y'' + 2cy' + \omega_0^2 = 0$$

18/20

Since the solution is overdamped, $c > \omega_0$. Hence, the characteristic polynomial would be

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0$$

$$\hookrightarrow \lambda_{1,2} = \frac{-2c \pm \sqrt{4c^2 - 4\omega_0^2}}{2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

since $c > \omega_0$, λ_1 and λ_2 are real, distinct solutions. Hence, the general (overdamped) solution to the differential equation would be

$$y(t) = C_1 e^{(-c + \sqrt{c^2 - \omega_0^2})t} + C_2 e^{(-c - \sqrt{c^2 - \omega_0^2})t}$$ ✓

One thing to note is that since $c > \omega_0$, $\lambda_1 = -c + \sqrt{c^2 - \omega_0^2}$ and $\lambda_2 = -c - \sqrt{c^2 - \omega_0^2}$ are both less than 0, since $\sqrt{c^2 - \omega_0^2} < c$. Hence, both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ cannot equal 0 for any t. Therefore, the only way for $y(t) = 0$ is for:

1. If C_1 and $C_2 = 0$, then there is no motion, so C_1 and C_2 cannot both be 0.
2. ✓ If $C_1 = 0$ and $C_2 \neq 0$ or vice versa, then because $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ cannot equal 0, the x-axis (time axis) is NOT crossed.
3. needs elaboration. If $C_1 \neq 0$ and $C_2 \neq 0$, then the only way to cross the time axis is such that for some t_0, where $y(t_0) = 0$,

$$C_1 e^{\lambda_1 t_0} + C_2 e^{\lambda_2 t_0} = 0$$

$$\underbrace{e^{\lambda_1 t_0}}_{e^{\lambda_1 t_0} \neq 0} \left[C_1 + \underbrace{C_2 e^{(\lambda_2 - \lambda_1)}}_{} \right] = 0$$

$$C_1 + C_2 e^{(\lambda_1 - \lambda_2)} = 0$$

$$\boxed{C_1 = -C_2 e^{(\lambda_1 - \lambda_2)}}$$ ✓

Since $y(t)$ only crosses the time axis for 1 t_0 such that $C_1 e^{\lambda_1 t_0} = -C_2 e^{\lambda_2 t_0}$ or $C_1 = -C_2 e^{(\lambda_1 - \lambda_2)}$, $y(t)$ only crosses the time axis once for any initial condition.

2. (20 points) Find a particular solution of the following equation.

$$y'' + 4y' + 4y = e^{-2t} + \sin(2t).$$

Using Method of Undetermined Coefficients:

$y_p = y_1 + y_2$, where y_p is a particular solution to the 2nd ODE above.

Say $y_1(t) = ae^{-2t}$ and $y_2(t) = a\cos(2t) + b\sin(2t)$

Solving for y_1:

$y_1' = -2ae^{-2t}$

$y_1'' = 4ae^{-2t}$

$$y_1'' + 4y_1' + 4y_1 = e^{-2t}$$

$$4ae^{-2t} + 4(-2ae^{-2t}) + 4(ae^{-2t}) = \cancel{4ae^{-2t}} \cancel{-8ae^{-2t}} + \cancel{4ae^{-2t}} \neq e^{-2t}$$

$\Rightarrow$ Instead, say $y_1 = ate^{-2t}$.

$y_1' = ae^{-2t} - 2ate^{-2t}$

$y_1'' = -2ae^{-2t} - 2ae^{-2t} + 4ate^{-2t}$

$= -4ae^{-2t} + 4ate^{-2t}$

$$\Rightarrow \quad y_1'' + 4y_1' + 4y_1 = e^{-2t}$$

$$-4ae^{-2t} + 4ate^{-2t} + 4(ae^{-2t} - 2ate^{-2t}) + 4ate^{-2t} = e^{-2t}$$

$$\cancel{-4ae^{-2t}} + \cancel{4ate^{-2t}} + \cancel{4ae^{-2t}} - 8\cancel{ate^{-2t}} + \cancel{4ate^{-2t}} = e^{-2t}$$

$\Rightarrow$ Instead, say $y_1 = at^2e^{-2t}$

$y_1' = 2ate^{-2t} - 2at^2e^{-2t}$

$y_1'' = 2ae^{-2t} - 4ate^{-2t} - 4ate^{-2t} + 4at^2e^{-2t}$

$= 2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t}$

$$\Rightarrow \quad y_1'' + 4y_1' + 4y_1 = e^{-2t}$$

$$2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t} + 4(2ate^{-2t} - 2at^2e^{-2t}) + 4(at^2e^{-2t}) = e^{-2t}$$

$$= 2ae^{-2t} \cancel{-8ate^{-2t}} + \cancel{4at^2e^{-2t}} + \cancel{8ate^{-2t}} \cancel{-8at^2e^{-2t}} + \cancel{4at^2e^{-2t}} = e^{-2t}$$

$$2ae^{-2t} = e^{-2t} \Rightarrow a = \frac{1}{2}$$

$y_1 = \frac{t^2}{2}e^{-2t}$

KT

Solving for y_2:

$y_2'' + 4y_2' + 4y_2 = \sin(2t)$

$y_2 = a\cos(2t) + b\sin(2t)$

$y_2' = -2a\sin(2t) + 2b\cos(2t)$

$y_2'' = -4a\cos(2t) - 4b\sin(2t)$

$\Rightarrow -\cancel{4a\cos(2t)} - \cancel{4b\sin(2t)} + 4(-2a\sin(2t) + 2b\cos(2t)) + \cancel{4a\cos 2t} + \cancel{4b\sin t}$

$= -8a\sin(2t) + 8b\cos(2t) = \sin(2t)$

$\Downarrow$

$b = 0, \quad -8a\sin(2t) = \sin 2t)$

$a = -\frac{1}{8}$

$y_2 = -\frac{1}{8}\cos(2t)$

$\boxed{\therefore\ y_p(t) = \frac{t^2}{2}e^{-2t} - \frac{1}{8}\cos(2t)}$

KT

3. (20 points) Find a particular solution of the following second order ODE.

$$y'' - 2y' + y = e^t.$$

Using ~~Variation~~ of Params

Solve associated hom. eq: $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda = 1$

Hence, fundamental solution set of assoc. hom. eq. is:

$\{\underset{y_1}{e^t}, \underset{y_2}{te^t}\}$

$\Rightarrow y_p = v_1 y_1 + v_2 y_2 = v_1 e^t + v_2(te^t)$

$y_p' = (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2') = v_1 y_1' + v_2 y_2'$

$v_1' y + v_2' y_2 = 0$

$y_p'' = v_1' y_1' + v_2' y_2' + v_1 y'' + v_2 y''$

Substitute into 2^{nd} ODE:

$v_1' y_1' + v_2' y_2' + v_1 y'' + v_2 y'' - 2v_1 y_1' - 2v_2 y_2' + v_1 y_1 + v_2 y_2$

$= v_1' e^t + v_2'(e^t + te^t) + v_1(e^t) + v_2(e^t + e^t + te^t) - 2v_1(e^t) - 2v_2(e^t + te^t)$
$+ v_1(e^t) + v_2(te^t)$

$= v_1' e^t + v_2'(e^t + te^t) + \cancel{2v_1 e^t} + \cancel{2v_2 e^t} + \cancel{2v_2 te^t} - \cancel{2v_1 e^t} - \cancel{2v_2 e^t} - \cancel{2v_2 te^t}$

$\Rightarrow v_1' e^t + v_2'(e^t + te^t) = e^t$

$\begin{cases} v_1' e^t + v_2'(te^t + e^t) = e^t \\ v_1' e^t + v_2'(te^t) = 0 \end{cases} \Longrightarrow - \begin{cases} v_1' + v_2'(t+1) = 1 \\ v_1' + v_2' t = 0 \end{cases}$

since $e^t \neq 0$

$v_2' = 1 \rightarrow v_2 = t$

$v_1' + t = 0 \quad v_1' = -t \rightarrow v_1 = -\frac{t^2}{2}$

$\therefore y_p = -\frac{t^2}{2} e^t + t^2 e^t$

$= \boxed{\frac{t^2}{2} e^t}$

KT

4. (20 points) Find the general solution of the system for the given matrix.

$$\mathbf{y}' = A\mathbf{y},$$

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}.$$

$$A-\lambda I = \begin{pmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(6-\lambda)+4 = \lambda^2 - 2\lambda - 6\lambda + 12 + 4$$
$$= \lambda^2 - 8\lambda + 16 = 0$$
$$= (\lambda-4)2$$

$\boxed{\lambda_1 = 4:}$ $A-4I = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\vec{v}_1 = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\hookrightarrow v_{11} - 2v_{12} = 0 \qquad v_{11} = 2v_{12} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$(A-4I)\vec{v}_2 = \vec{v}_1 \Rightarrow \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} -2 & 4 & 2 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$v_{21} - 2v_{22} = -1 \qquad v_{21} = 2v_{22} - 1 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore \boxed{\vec{y}(t) = e^{4t}\left((C_1 + C_2 t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)}$$

KT

5. (20 points) Solve the initial value problem of linear systems for the given matrix.

$$\mathbf{y}' = A\mathbf{y},$$

$$A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix},$$

$$\mathbf{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

$$A - \lambda I = \begin{pmatrix} -5-\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix} \Rightarrow \det(A-\lambda I) = (-5-\lambda)(-2-\lambda) + 2$$

$$= \lambda^2 + 5\lambda + 2\lambda + 10 + 2$$

$$= \lambda^2 + 7\lambda + 12 = 0$$

$$(\lambda+3)(\lambda+4) = 0$$

$$\lambda_{1,2} = -3, -4$$

$\boxed{\lambda_1 = -3}$ $A + 3I = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}$

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2v_{11} - v_{12} = 0 \Rightarrow v_{12} = 2v_{11}$$

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\boxed{\lambda_2 = -4}$ $A + 4I = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$-v_{21} + v_{22} = 0 \Rightarrow v_{21} = v_{22}$$

$$\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{y}(t) = c_1 e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} c_1 e^{-3t} + c_2 e^{-4t} \\ 2c_1 e^{-3t} + c_2 e^{-4t} \end{pmatrix}$$

$$\vec{y}(0) = \begin{pmatrix} c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow c_1 + c_2 = 0 \to c_1 = -c_2$$

$$2c_1 + c_2 = -1 \to -2c_2 + c_2 = -1$$

$$-c_2 = -1$$

$$\underline{c_1 = -1} \leftarrow \underline{c_2 = 1}$$

$$\boxed{\therefore \vec{y}(t) = -e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$