Math 33B-1, Winter 2019

MIDTERM 1

Address all parts of each question and show your work to receive full credit. Write clearly and box/circle/highlight your final answer. If you need scratch paper or additional space to write your answers, raise your hand and your instructor will bring some to you.

No notes, textbooks, calculators, computers, cell phones or mobile devices are allowed.

By signing below you acknowledge that you have read the above instructions and you will abide by the University Honor Code.

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STUDENT ID: 305 100 413

DISCUSSION SECTION: 1F

SIGNATURE: Nathan Yuen

Problem	Pts Possible	Pts Earned
1	5	4
2	10	10
3	9	7
4	8	8
5	10	9
6	8	8

Total: 46 /50

Points earned: ______

(5 pts) 1. Circle 'true' or 'false' for each of the following statements. You do not need to justify your answers.

(a) The differential equation $y' + y + t = 0$ is homogeneous.

true false

(b) The differential equation $y' = e^{x+y}$ is separable.

true false

(c) The initial value problem $x' = \sqrt{x}$, $x(2) = 3$ is guaranteed a unique solution by the uniqueness theorem.

true false

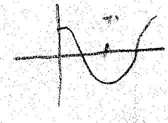
(d) The differential equation $x' - x = \sin(t)$ is in normal form.

true false

(e) The differential equation $y' + 2y\sin(t) = e^t$ is linear.

true false

 Points earned: ____________



(10 pts) 2. Use separation of variables to solve the following initial value problems.

(a) $2y' = \dfrac{\sin(x)}{y}$, $\quad y\left(\dfrac{\pi}{2}\right) = -2$

$2y'y = \sin x$

$2\, y\,dy = \sin x\, dx$

$2\int y\,dy = \int \sin x\, dx$

$y^2 = -\cos x + C$

$y^2 = -\cos x + c$

$y = \pm\sqrt{-\cos x + c}$

$y(\frac{\pi}{2}) = -2$:

$-2 = \pm\sqrt{-\cos(\frac{\pi}{2}) + c}$

$-2 = \pm\sqrt{c}$

$C = 4$, take the negative root

$\boxed{y = -\sqrt{-\cos(x) + 4}}$

(b) $y' - 2y = e^x y$, $\quad y(0) = 3e$

$y' = e^x y + 2y$

$\frac{y'}{y} = e^x + 2$

$\frac{dy}{y} = (e^x + 2)dx$

$\int \frac{dy}{y} = \int (e^x + 2)\, dx$

$\ln|y| = e^x + 2x + C$

$y = Ae^{e^x}e^{2x}$

where $A = e^C$

$y(0) = 3e$:

$3e = Ae^{e^0}e^0$

$3e = Ae^1(1)$

$A = 3$

$\boxed{y = 3e^{e^x}e^{2x}}$

Points earned: 10

(9 pts) 3. **Existence-uniqueness**

(a) (5 points) Consider the following initial value problem:

$$y' = 2t(y-2)^{1/3}, \qquad y(1) = 2$$

Show that the existence theorem **guarantees** a solution to this problem. Then explain why this solution is **not guaranteed** to be unique, using the uniqueness theorem to support your argument.

$y' = f(t,y)$; $f(t,y)$ is ~~defined~~ continuous ~~for~~ all t, y

Thus pick any R_1^* containing $(1,2)$ such as $0 < t < 2$, $1 < y < 3$. Since f is defined on all of R_1 there is a guaranteed solution to the problem.

* R_1, R_2 rectangles

$\frac{\partial f}{\partial y} = \frac{2t}{3(y-2)^{2/3}}$ For the uniqueness thm to hold, both f and $\frac{\partial f}{\partial y}$ must be defined on some R_2^* around $(1,2)$. However, $\frac{\partial f}{\partial y}$ is undefined at $y=2$, so it is impossible to pick such R_2. Thus the uniqueness theorem does not guarantee a unique solution.

(b) (4 points) Show that the uniqueness theorem **guarantees** that a unique solution exists to the following initial value problem:

$$y' = x^2 \ln(y), \qquad y(1) = 2$$

$f(x,y) = y'$

$\frac{\partial f}{\partial y} = \frac{x^2}{y}$. Both f and $\frac{\partial f}{\partial y}$ are ~~defined~~ cont. when $y \in (0, \infty)$. Thus pick a rectangle R satisfying

$$\begin{cases} y \in (0, \infty) \\ \text{contains } (1,2) \end{cases}$$

One such R is $0 < x < 2$, $1 < y < 3$. Since both f and $\frac{\partial f}{\partial y}$ are defined on R, there is a guaranteed unique solution at $(1,2)$ by the uniqueness thm.

Points earned: 7

(8 pts) 4. Consider the following differential equation

$$ty' + 3y = t^2$$

(a) (4 points) Verify that the general solution to this equation is $y(t) = \frac{C}{t^3} + \frac{1}{5}t^2$, where C is an arbitrary constant (do not solve the differential equation).

$y' = -3Ct^{-4} + \frac{2}{5}t$ Substitute into left-hand side:

$t(-3Ct^{-4} + \frac{2}{5}t) + 3(Ct^{-3} + \frac{1}{5}t^2)$

$-3Ct^{-3} + \frac{2}{5}t^2 + 3Ct^{-3} + \frac{3}{5}t^2$

$= t^2$ which equals the right-hand side.

(b) (2 points) Using the general solution provided in part (a), find the particular solution to the initial value problem

$$ty' + 3y = t^2, \qquad y(-1) = -\frac{9}{5}$$

Substitute into $y(t)$:

$-\frac{9}{5} = \frac{C}{(-1)^3} + \frac{1}{5}(-1)^2$

$-\frac{9}{5} = -C + \frac{1}{5}$

$-2 = -C$

$C = 2$

$\boxed{y = \frac{2}{t^3} + \frac{1}{5}t^2}$ ✓

(c) (2 points) What is the interval of existence for the particular solution you found in part (b)?

In the general solution, y is defined everywhere except at $t=0$. However, the interval must contain our initial t-value, -1. Thus the interval:

$\boxed{(-\infty, 0)}$ ✓

Points earned: 8

(10 pts) 5. A tank initially contains 100 gallons of pure water. Brine with 2 pounds (lbs) of salt per gallon flows into the tank at 5 gallons per minute. The well-mixed solution runs out of the tank, also at 5 gallons per minute.

(a) (4 points) Derive the following differential equation for $y(t)$, the amount of salt (in lbs) in the tank after t minutes:

$$\frac{dy}{dt} = 10 - \frac{y}{20}.$$

Then, explain why the initial condition is $y(0) = 0$. (-1)

$\frac{dy}{dt}$ = Rate in − rate out (of salt)

Rate in: 2 lbs/gal × 5 gal/min = 10 lbs/min

Rate out: concentration × 5 gal/min = $\frac{y}{100} \times 5 = \frac{y}{20}$ lbs/min

$\frac{dy}{dt} = 10 - \frac{y}{20}$

(b) (5 points) Solve the initial value problem from part (a).

$\frac{dy}{dt} + \frac{y}{20} = 10$ $\qquad \frac{dy}{dt} + f(t)\,y = 10$

Integrating factor: $e^{\int \frac{1}{20} dt} = e^{t/20}$

$e^{t/20}\left(\frac{dy}{dt} + \frac{y}{20}\right) = 10e^{t/20}$

$(ye^{t/20})' = 10e^{t/20}$

$\int (ye^{t/20})' = \int 10e^{t/20}\,dt$

$ye^{t/20} = 200e^{t/20} + C$

$y = 200 + Ce^{-t/20}$

Initial condition: $y(0) = 0$:

$0 = 200 + Ce^{0} \rightarrow C = -200$

$\boxed{y = 200 - 200e^{-t/20}}$

(c) (1 point) How much salt is in the tank after 40 minutes? You do not need to evaluate your answer numerically. For example, an answer of $\ln(\pi)$ should be left like that.

Using our solution: $y(40) = 200 - 200e^{-(40)/20}$

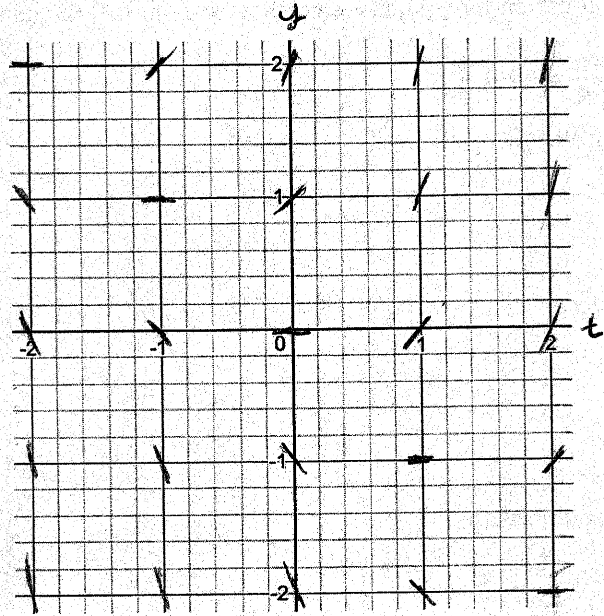
$y(40) = 200 - 200e^{-2}$

$= \boxed{200\left(1 - \frac{1}{e^2}\right) \text{ lbs}}$

Points earned: ______________

(8 pts) **6. Direction fields**

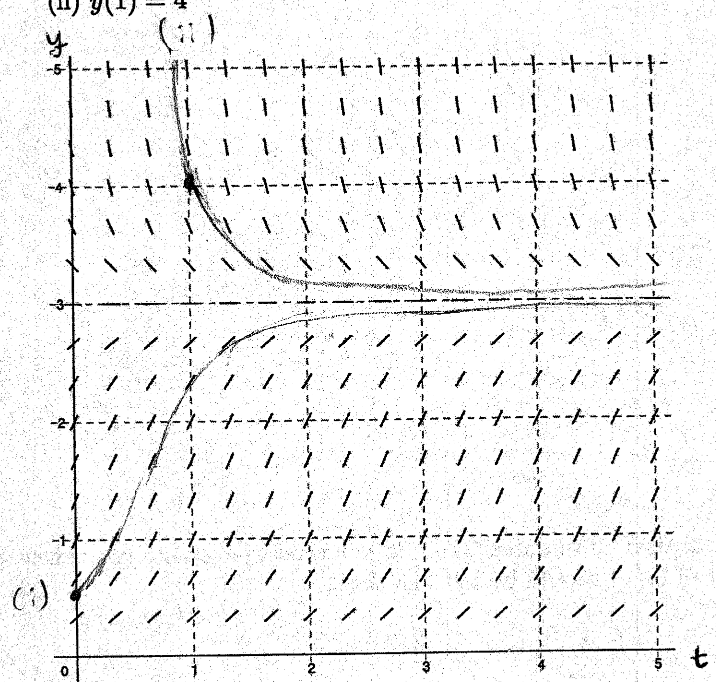
(a) (4 points) On the axes below, sketch the direction field for the differential equation $y' = t + y$. You only need to indicate the slopes at integer ordered pairs.



✓

(b) (4 points) The image below is the direction field for the differential equation $y' = y(3 - y)$. Sketch solution curves on the direction field for the following initial conditions:

(i) $y(0) = 0.5$
(ii) $y(1) = 4$



✓

Points earned: 8