Math 33B, Lec 2
Spring 2016
Exam 2
5/9/16
Time Limit: 50 Minutes

Name (Print): Jerin Tomy

Name (Sign): [signature]

Discussion Section: 2E

This exam contains 6 pages, including this cover page and 5 problems.

You may *not* use books, notes, or any calculator on this exam.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Problem	Points	Score
1	10	10
2	10	10
3	10	10
4	10	10
5	10	10
Total:	50	50

1. (10 points) (a) Find the general solution to the differential equation

$$y'' - 8y' + 16y = 0.$$

$(\lambda-4)(\lambda-4)$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda-4)^2 = 0$

$y_1 = e^{4t}$ $y_2 = te^{4t}$

$\boxed{y(t) = C_1 e^{4t} + C_2 t e^{4t}}$

5

(b) Find the general solution to the differential equation

$$y'' - 8y' + 20y = 0.$$

64 80   80 64 16

$\lambda^2 - 8\lambda + 20 = 0$

$z(t) = e^{(4+2i)t}$ $\lambda = \frac{8 \pm \sqrt{64-(4)(1)(20)}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$

$z(t) = e^{4t} e^{2it} = e^{4t}(\cos 2t + i \sin 2t)$

$\boxed{y(t) = C_1 e^{4t}\cos 2t + C_2 e^{4t} \sin 2t}$

5

2. Let $y_1(t) = t$ and $y_2(t) = t^3$.

(a) (3 points) Calculate the Wronskian of y_1 and y_2.

$y_1(t) = t$ $y_2(t) = t^3$

$y_1'(t) = 1$ $y_2'(t) = 3t^2$

$y_1''(t) = 0$ $y_2''(t) = 6t$

$\det \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 3t^3 - t^3 = \boxed{2t^3}$

(b) (7 points) Show that y_1 and y_2 are a fundamental set of solutions to the equation

$$t^2y'' - 3ty' + 3y = 0.$$

$\frac{y_2}{y_1} = \frac{t^3}{t} = t^2$ non constant multiple, so linearly independent

Check y_1: $-3t(1) + 3(t) = 0$

$0 = 0$

Check y_2: $t^2(6t) - 3t(3t^2) + 3(t^3)$

$6t^3 - 9t^3 + 3t^3 = 0$

$0 = 0$

Since y_1 & y_2 are both solutions to the second order equation and are linearly independent, they form a fundamental set of solutions to the equation

3. (10 points) Find the solution to the initial-value problem

$$y'' + y' - 12y = 0, \; y(0) = 1, \; y'(0) = 10.$$

$\lambda^2 + \lambda - 12 = 0$

$(\lambda + 4)(\lambda - 3)$

$\lambda^2 - 3\lambda + 4\lambda - 12$

$\lambda^2 + \lambda - 12$

$y(t) = C_1 e^{-4t} + C_2 e^{3t}$

$y(0) = C_1 + C_2 = 1$

$y'(t) = -4C_1 e^{-4t} + 3C_2 e^{3t}$

$y'(0) = -4C_1 + 3C_2 = 10$

$y(t) = -e^{-4t} + 2e^{3t}$

10

$C_1 + C_2 = 1$

$-4C_1 + 3C_2 = 10$

$4C_1 + 4C_2 = 4$

$7C_2 = 14$

$C_2 = 2$

$C_1 = -1$

4. (10 points) Find the solution to the initial-value problem

$$y'' + y' - 12y = 6e^{2t} - 144t,\ y(0) = 1,\ y'(0) = 7.$$

$y = y_p + y_h$

Break into two parts

$y'' + y' - 12y = 6e^{2t}$

try $y = ae^{2t}$

$y' = 2ae^{2t}$

$y'' = 4ae^{2t}$

$4ae^{2t} + 2ae^{2t} - 12ae^{2t} = 6e^{2t}$

$-6ae^{2t} = 6e^{2t}$, $a = -1$

$y = -e^{2t}$

$y'' + y' - 12y = -144t$

try $y = at + b$

$y' = a$

$y'' = 0$

$0 + a - 12(at + b) = -144t$

$a - 12at - 12b$

$-12a = -144$

$a = 12$

$a - 12b = 0$

$b = 1$

$y = 12t + 1$

$y_p = -e^{2t} + 12t + 1$

From previous problem

y_h: $C_1 e^{-4t} + C_2 e^{3t}$

$y(t) = C_1 e^{-4t} + C_2 e^{3t} - e^{2t} + 12t + 1$

$$\boxed{y(t) = \frac{6}{7}e^{-4t} + \frac{1}{7}e^{3t} - e^{2t} + 12t + 1}$$

$y(0) = C_1 + C_2 - 1 + 1 = 1$

$C_1 + C_2 = 1$

$-4C_1 + 3C_2 = -3$

$y'(t) = -4C_1 e^{-4t} + 3C_2 e^{3t} - 2e^{2t} + 12$

$y'(0) = -4C_1 + 3C_2 - 2 + 12 = 7$

$C_1 + C_2 = 1$

$-4C_1 + 3C_2 = -3$

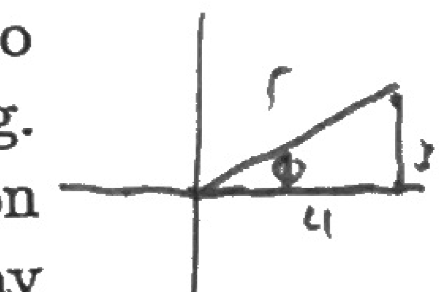
$4C_1 + 4C_2 = 4$

$C_2 = \frac{1}{7}$

$C_1 = \frac{6}{7}$

5. A mass of 5 kg is attached to a large spring with a spring constant of $k = 20\ \text{kg/s}^2$.

(a) (7 points) The system is then stretched 4 m from the spring-mass equilibrium and set to oscillating with an initial velocity of 6 m/s. Assume that it oscillates without damping. Write the differential equation describing the motion of the system, and use the solution of the equation to find the frequency, amplitude, and phase of the vibration. (You may leave your phase in terms of arctan.)



same direction (assume positive) $\phi = \tan^{-1}\left(\frac{3}{4}\right)$

$5y'' + 20y = 0$ $\quad y(0) = 4,\ y'(0) = 6$

$y'' + 4y = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$y(t) = C_1 \cos 2t + C_2 \sin 2t$

$y(0) = C_1 = 4$

$y'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t$

$y'(0) = 2C_2 = 6$

$C_2 = 3$

10

$y(t) = 4\cos 2t + 3\sin 2t$ ✓

$y(t) = 5\left(\frac{4}{5}\cos 2t + \frac{3}{5}\sin 2t\right)$

$y(t) = 5(\cos\phi \cos 2t + \sin\phi \sin 2t)$

$y(t) = 5\cos(2t - \phi)$

$\boxed{y(t) = 5\cos\left(2t - \tan^{-1}\left(\frac{3}{4}\right)\right)}$

Amplitude: 5 m ✓

phase: $\tan^{-1}\left(\frac{3}{4}\right)$ rad ✓

Frequency: 2 rad/s ✓

(b) (3 points) Now suppose the system is placed in a viscous medium that supplies a damping constant that gives the system critical damping. Find the value of the damping constant μ for which the system is critically damped.

let $\alpha = \frac{\mu}{5}$

$5y'' + \mu y + 20y = 0$

$\boxed{\mu = 20\ \text{kg/s}}$

$y'' + \alpha y + 4y = 0$

$\lambda^2 + \alpha\lambda + 4 = 0$

$\alpha = 4 \quad c \quad \omega_0^2$ $\quad c =$

$(\lambda^2 + 4\lambda + 4)$

single, repeated roots, critically damped $\rightarrow (\lambda + 2)^2$