Math 33B, Lec 2
Spring 2016
Exam 1
4-18-16
Time Limit: 50 Minutes

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Discussion Section: 2A

This exam contains 6 pages, including this cover page and 5 problems.

You may *not* use books, notes, or any calculator on this exam.

There will be no partial credit given for problems 1 and 2. For the other problems, partial credit will only be awarded to answers for which an explanation and/or work is shown.

Problem	Points	Score
1	10	8
2	10	8
3	10	10
4	10	10
5	10	10
Total:	50	46

1. (10 points) There are five statements below. Decide which are true and which are false. On the left of each, write the full word "TRUE" or "FALSE." There is no partial credit on this problem.

FALSE ________ (i) The function $G(x,y) = \ln(x^2) - \ln(y) + e^{\frac{x}{y}}$ is homogeneous.

$\ln((tx)^2) - \ln(ty) + e^{\frac{tx}{ty}}$

$\ln\frac{(tx)^2}{(ty)} + e^{\frac{x}{y}} = \ln\frac{t^2x^2}{\cancel{t}y} + e^{\frac{x}{y}}$

X TRUE ________ (ii) The ODE $y' = \dfrac{x^3y}{y'}$ has a unique solution satisfying $y(1) = 1$.

$y'^2 = x^3y$

$y' = \sqrt{x^3y}$

$f(x,y) = \sqrt{x^3y} = (x^3y)^{1/2}$

continuous for $x, y \geq 0$

$\partial f/\partial y = \frac{1}{2}(x^3y)^{-1/2}(x^3) = \dfrac{x^3}{2\sqrt{x^3y}}$

continuous for $x, y > 0$

TRUE ________ (iii) The equation $4xy^3dx + 6x^2y^2dy = 0$ is an exact differential equation.

$\frac{\partial P}{\partial y} = 12xy^2 \quad \Big| \quad \frac{\partial Q}{\partial x} = 12xy^2$ ✓

FALSE ________ (iv) The function $y(t) = e^{-t}$ is a solution to the ODE $y' = -ty$.

$y' = -e^{-t}$

$y' = -ty = -te^{-t} \neq -e^{-t}$

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TRUE ________ (v) The function $\mu(y) = \dfrac{1}{y^2}$ is an integrating factor for the ODE

$$\frac{(y^2 + 2xy)dx}{y^2} - \frac{x^2dy}{y^2} = 0.$$

$(1 + \frac{2x}{y})dx + (\frac{x^2}{y^2})dy$

$(1 + 2xy^{-1})$

$(-x^2y^{-2})$

$\frac{\partial P}{\partial y} = -2xy^{-2} \quad \Big| \quad \frac{\partial Q}{\partial x} = -2xy^{-2}$

2. (10 points) There are five multiple choice questions below. Each question has one correct answer. On the left of each question, write the letter of the correct answer. There is no partial credit on this problem.

B (i) The function $G(x, y) = xe^{\frac{x^2}{y^2}} + y$ is homogenous of degree:

(A) 0 (B) 1 (C) 2 (D) 3 (E) not homogeneous

$txe^{\frac{x^2}{y^2}} + ty$

?

~~B~~ (ii) The ODE $y' = y^{1/3}$ does *NOT* have a unique solution satisfying the initial condition:

~~(A)~~ $y(1) = -2$ (B) $y(0) = -1$ ~~(C) $y(1) = 0$~~ ~~(D) $y(1) = 1$~~ (E) $y(0) = 2$

$f(x,y) = y^{1/3}$ $\partial f/\partial y = \frac{1}{3}y^{-2/3} = \frac{1}{3y^{2/3}}$ $y \neq 0$

$\frac{dy}{dx} = y^{1/3}$ $\int y^{-1/3}dy = \int dx$ $\frac{3y^{2/3}}{2} = x + c$ $y^{2/3} = \frac{2}{3}(x+c)$ $y = (\frac{2}{3}(x+c))^{3/2}$

$3(-1)^{2/3} = 0 + c$

E (iii) The differential equation $y' = \dfrac{\sin(t + y)}{y}$ is:

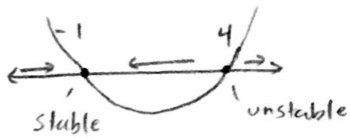
~~(A)~~ separable (B) linear ~~(C)~~ exact ~~(D)~~ homogeneous (E) none of these options

$\frac{dy}{dy}$ $\sin(t+y)\,dt$

C (iv) Consider $y' = (y - 4)(y + 1)$. The equilibria 4 and -1 have the property that:

(A) both are locally stable (B) 4 is locally stable, -1 is unstable

(C) 4 is unstable, -1 is locally stable (D) both are unstable (E) -1 is not an equilibrium



C (v) Which of the following does not satisfy $y' = y^2 + y$:

$0 = 1^2 + 1 = 2$

$1^2 + 1$

(A) $y(t) = 0$ for all t (B) $y(t) = \dfrac{5e^t}{1 - 5e^t}$ (C) $y(t) = 1$ for all t

$(-1)^2 + (-1) = 0$

(D) $y(t) = \dfrac{1}{e^{-t} - 1}$ (E) $y(t) = -1$ for all t ✓

3. (10 points) (a) Find the general solution to the separable differential equation $y' = 2xy + x^2y$.

$$y' = 2xy + x^2 y$$

$$y' = y(2x + x^2)$$

$$\frac{dy}{dx} = y(2x + x^2)$$

$$\int \frac{dy}{y} = \int (2x + x^2)\,dx$$

$$e^{\ln|y|} = e^{\left(x^2 + \frac{x^3}{3} + c\right)}$$

$$\boxed{y = Ce^{\left(x^2 + \frac{x^3}{3}\right)}}$$

(b) Find the particular solution to the initial-value problem $y' = 2xy + x^2y + x^2e^{x^2}$, $y(0) = 0$.

$$y' = 2xy + x^2y + x^2e^{x^2}, \quad y(0) = 0$$

$$y' = y(2x + x^2) + x^2 e^{x^2} \qquad \text{linear}$$

$$a(x) = 2x + x^2$$

$$f(x) = x^2 e^{x^2}$$

$$u(x) = e^{-\int (2x + x^2)dx} = e^{-\left(x^2 + \frac{x^3}{3}\right)}$$

$$e^{-\left(x^2 + \frac{x^3}{3}\right)} y = \int e^{-x^2 - \frac{x^3}{3}} x^2 e^{x^2}\,dx$$

$$= \int e^{-\frac{x^3}{3}} x^2\,dx \qquad u = -\frac{x^3}{3}$$

$$du = -\frac{3x^2}{3} = -x^2dx$$

$$= -\int e^u\,du$$

$$e^{-\left(x^2 + \frac{x^3}{3}\right)} y = -e^{-x^3/3} + C$$

$$y = -\frac{e^{-x^3/3}}{e^{-x^2 - \frac{x^3}{3}}} + Ce^{\left(x^2 + \frac{x^3}{3}\right)}$$

$$y = -e^{x^2} + Ce^{\left(x^2 + \frac{x^3}{3}\right)}$$

$$y(0) = 0$$

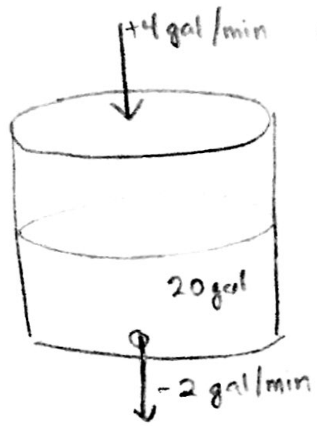
$$0 = -1 + C \rightarrow C = 1$$

$$\boxed{y = -e^{x^2} + e^{\left(x^2 + \frac{x^3}{3}\right)}}$$

?

10

4. (10 points) A 50-gal tank initially contains 20 gal of pure water. Salt-water solution containing 0.5 lb of salt for each gallon of water begins entering the tank at a rate of 4 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 gal/min. What is the salt content (lb) in the tank at the precise moment that the tank is full of salt-water solution?



$$c(t) = \frac{x(t)}{v(t)} = \frac{x(t)}{20+2t}$$

$$v(t) = 20 + 4t - 2t = 20 + 2t$$

$$x' = \text{rate in} - \text{rate out}$$

$$= \left(4\,\frac{gal}{min}\right)\left(0.5\,\frac{lb}{gal}\right) - \left(2\,\frac{gal}{min}\right)\left(\frac{x}{20+2t}\,\frac{lb}{gal}\right)$$

$$x' = 2 - \frac{x}{10+t} \quad (\text{linear})$$

$$x' = -\frac{1}{10+t}x + 2 \qquad a(t) = -\frac{1}{10+t}$$

$$f(t) = 2$$

$$u(t) = e^{-\int -\frac{1}{10+t}dt} = e^{\ln|10+t|} = 10+t$$

$$(10+t)x = \int (10+t)(2)\,dt + C$$

$$= 2\int (10+t)\,dt + C$$

$$(10+t)x = 2\left(10t + \frac{t^2}{2}\right) + C$$

$$= 2t\left(10 + \frac{t}{2}\right)$$

gen. soln.

$$\boxed{x = \frac{2t\left(10+\frac{t}{2}\right)}{10+t} + \frac{C}{10+t}}$$

$$x(0) = 0$$

$$0 = \frac{0}{10} + \frac{C}{10} \to C = 0$$

partic. soln.

$$\boxed{x = \frac{2t\left(10+\frac{t}{2}\right)}{10+t}}$$

The tank is full when

$$V(t) = 50 = 20 + 2t$$

$$30 = 2t \to t = 15$$

$$X(15) = \frac{2(15)\left(10 + \frac{15}{2}\right)}{10+15}$$

$$= \frac{30}{25}\left(\frac{20+15}{2}\right)$$

$$= \frac{\cancel{30}^{15}}{25}\cdot\frac{35}{\cancel{2}} = \frac{\cancel{15}^{3}}{\cancel{25}_{5}}\cancel{35}^{7} = \boxed{21 \text{ lbs}}$$

5. (10 points) Find the general solution to the exact differential equation

$$(x^2 + y^2 \sin(x))dx + 2y(1 - \cos(x))dy = 0.$$

$(x^2 + y^2 \sin x)dx + 2y(1 - \cos x)dy = 0$ $\qquad 2y - 2y\cos x$

$\frac{\partial P}{\partial y} = 2y\sin x \quad \Big| \quad \frac{\partial Q}{\partial x} = 2y\sin x \quad$ exact ✓

$F(x,y) = \int (x^2 + y^2 \sin x)\, dx + \phi(y)$

$= \frac{x^3}{3} - y^2 \cos x + \phi(y)$

$\partial F/\partial y = -2y\cos x + \phi'(y) = 2y - 2y\cos x$

$\phi'(y) = 2y \rightarrow \phi(y) = y^2$

$$\boxed{F(x, y) = \frac{x^3}{3} - y^2 \cos x + y^2 = C}$$

10/10