19F-MATH33B-2 Midterm II

MAGGIE YAN

TOTAL POINTS

35.5 / 45

QUESTION 1

Problem 1 10 pts

1.1 1a 6 / 6

✓ **- 0 pts** Correct

1.2 1b 0 / 2

✓ **- 2 pts** Incorrect

1.3 1c 2 / 2

✓ **- 0 pts** Correct

QUESTION 2

2 Problem 2 10 / 10

✓ **- 0 pts** Correct

QUESTION 3

3 Problem 3 8 / 8

✓ **- 0 pts** Correct

QUESTION 4

Problem 4 12 pts

4.1 4a 1 / 3

+ 1 pts Identified one impossible pair with correct reasoning

+ 2 pts Identified both impossible pairs with correct reasoning

✓ **+ 1 pts Identified one possible pair**

+ 0 pts Incorrect

4.2 4b 7 / 7

✓ **- 0 pts** Correct

4.3 4c 1.5 / 2

✓ **- 0 pts** Correct

- 0.5 Point adjustment

Don't need to solve for C again. Already did that while finding IVP in part b) . Solution is [first element of y(t)] = etcos(3t).

QUESTION 5

5 Problem 5 0 / 5

+ 2 pts Correct idea

✓ **+ 0 pts Not attempted / Incorrect**

+ 1 pts Showed conservation of mass mathematically AND used that in the argument.

+ 2 pts Stated reason y must cross 0.

Math 33B
Differential Equations

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to **write your name, section** and **UID** in the space below. Additionally, write your **UID** at the **top-right corner of every page.**

Name: Maggie Yan
Student ID number: 805126123
Section: 2E
Number of extra pages:

Question	Points	Score
1	10	
2	10	
3	8	
4	12	
Total:	40	

Problem 5 (in the last page) is a bonus problem for 5*pts*.

Problem 1.

Find the general solution of the following differential equations

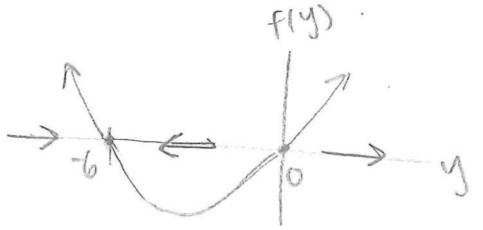
(a) [6pts.] Consider the autonomous differential equation

$$\frac{dy}{dx} = 6y + y^2,$$

Find the equilibrium points of the differential equation and characterize each equilibrium point as asymptotically stable or unstable. Justify your characterization of the equilibria using the first-derivative test.

$y' = f(y) = 6y + y^2 = y(6+y) = 0$

equilibrium points are at $y = 0$ and $y = -6$



$\boxed{\begin{array}{l} y = 0 \text{ is unstable} \\ y = -6 \text{ is stable} \end{array}}$

$f'(y) = 6 + 2y$

$f'(0) = 6 > 0 \Rightarrow$ unstable

$f'(-6) = -6 < 0 \Rightarrow$ stable

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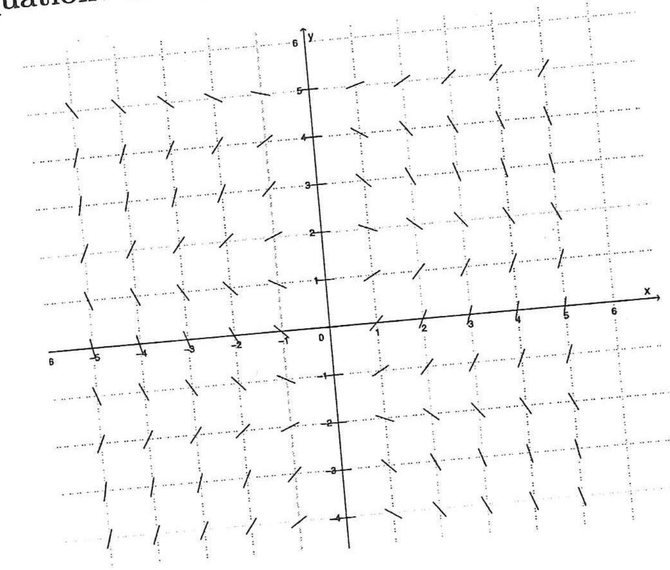
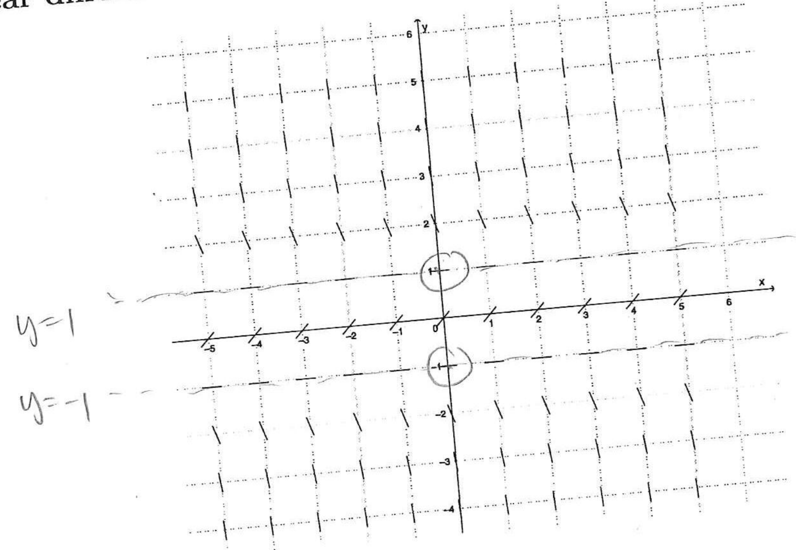
(b) [2pts.] Can the following be a direction field for a **first order autonomous differential equation**? If not, explain why.



yes

(c) [2pts.] Can the following figure be a direction field for a **autonomous first order linear differential equation**? If not, explain why.



no

there are 2 equilibrium points, meanin that this direction field isn't a linea diffeq.

Problem 2. *10pts.*
Find the general solution to the differential equation.

$$x''(t) + 2x'(t) + 2x(t) = e^{5t}. \tag{1}$$

① $x'' + 2x' + 2x = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-8}}{2}$

$= \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$

$y_h = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

② $y_p = Ae^{5t}$

$y_p' = 5Ae^{5t}$

$y_p'' = 25Ae^{5t}$

$25Ae^{5t} + 2(5Ae^{5t}) + 2(Ae^{5t}) = e^{5t}$

$Ae^{5t}(25+10+2) = e^{5t}$

$37A = 1 \Rightarrow A = \frac{1}{37}$

$y_p = \frac{1}{37}e^{5t}$

③ $\boxed{y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{1}{37}e^{5t}}$

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Problem 3. *8pts.*
Consider the equation

$$my''(t) + \mu y'(t) + ky(t) = 0 : \qquad (2)$$

Suppose $m = 4$ and $k = 10$. For what conditions of (non-negative) μ is the system i) Underdamped ii) Overdamped iii) Critically damped. You **do not** need to find the general solution of the the differential equation.

$4y'' + \mu y' + 10y = 0$

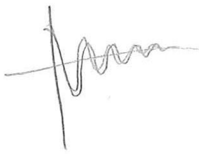
$4\lambda^2 + \mu\lambda + 10 = 0$

$\lambda = \dfrac{-\mu \pm \sqrt{\mu^2 - 4(40)}}{2(4)}$

i) underdamped

$b^2 - 4ac < 0$

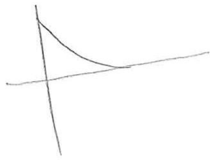
$\boxed{\mu < 4\sqrt{10}}$



ii) overdamped

$b^2 - 4ac > 0$

$\boxed{\mu > 4\sqrt{10}}$



iii) critically damped

$b^2 - 4ac = 0$

$\mu^2 = 4(4)(10)$

$\mu^2 = 160$

$\boxed{\mu = 4\sqrt{10}}$

Problem 4.

(a) [3pts.] Which one of these are possible pairs of solutions to a planar linear system:

$\lambda = -6, 6$

i) $\begin{bmatrix} 5e^{3t} \\ 10e^{3t} \end{bmatrix}, \begin{bmatrix} -2e^{4t} \\ -4e^{4t} \end{bmatrix}$ ii) $\begin{bmatrix} e^{-6t} \\ e^{-6t} \end{bmatrix}, \begin{bmatrix} -e^{6t} \\ e^{6t} \end{bmatrix}$ iii) $\begin{bmatrix} 9e^{it} \\ 9ie^{it} \end{bmatrix}, \begin{bmatrix} e^{-it} \\ ie^{-it} \end{bmatrix}$

$\lambda = 3, 4$

For each pair that is not possible, provide one or two lines of explanation stating why it is not a possible pair. For the pairs that can be solutions, you **do not need to** justify your answer.

iii) not possible because it has an imaginary component to both solutions

(b) [7pts.] Find the solution to the following IVP

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \mathbf{y}; \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A

$\det(A - \lambda I) = 0$

$\lambda^2 - T\lambda + D = 0$

$\lambda^2 - 2\lambda + 10 = 0$

$\lambda = \frac{2 \pm \sqrt{4-40}}{2}$

$= 1 \pm 3i$

$\lambda = 1+3i$: $\left[\begin{array}{cc|c} -1-3i & 1 & 0 \\ -10 & 1-3i & 0 \end{array}\right]$

$(1+3i)v_1 = v_2$

$\vec{v} = \begin{bmatrix} 1 \\ 1+3i \end{bmatrix}$

$e^{(1+3i)t} \begin{bmatrix} 1 \\ 1+3i \end{bmatrix} = e^t e^{3it} \begin{bmatrix} 1 \\ 1+3i \end{bmatrix}$

$e^t(\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ 1+3i \end{bmatrix}$

$= e^t \begin{bmatrix} \cos 3t + i\sin 3t \\ \cos 3t + 3i\cos 3t + i\sin 3t - 3\sin 3t \end{bmatrix}$

$= \underbrace{e^t \begin{bmatrix} \cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix}}_{Re} + i \underbrace{e^t \begin{bmatrix} \sin 3t \\ 3\cos 3t + \sin 3t \end{bmatrix}}_{Im}$

$y(t) = c_1 e^t \begin{bmatrix} \cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 3t \\ 3\cos 3t + \sin 3t \end{bmatrix}$

$y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1-0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3-0 \end{bmatrix}$

$\begin{cases} c_1 = 1 \\ c_1 + 3c_2 = 1 \end{cases} \Rightarrow c_2 = 0$

$$\boxed{y(t) = e^t \begin{bmatrix} \cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix}}$$

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(c) [2pts.] Find the solution to the following IVP for a second order equation **using** the solution of the linear system that you found in **part b)** of this problem.

$$x'' - 2x' + 10x = 0; \quad x(0) = 1, x'(0) = 1$$

$A = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \Rightarrow$ same as part b

$\begin{bmatrix} x \\ x' \end{bmatrix} = y(t) = c_1 e^t \begin{bmatrix} \cos 3t \\ \cos 3t - 3\sin 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 3t \\ 3\cos 3t + \sin 3t \end{bmatrix}$

$x(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t$

$x(0) = c_1 = 1$

$x'(t) = c_1 e^t (\cos 3t - 3\sin 3t) + c_2 e^t (3\cos 3t + \sin 3t)$

$x'(0) = c_1(1-0) + c_2(3+0) = 1 + 3c_2 = 1$

$3c_2 = 0 \Rightarrow c_2 = 0$

$\boxed{x(t) = e^t \cos 3t}$ → same solution as 2b when solving for coefficients

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Problem 5(Bonus problem) *5pts.*
Consider the IVP

$$\begin{aligned}
\dot{y}_1 &= -ay_1 + by_2 \\
\dot{y}_2 &= ay_1 - by_2 \\
y_1(0) &= y_{00} \;\; y_2(0) = y_{10}
\end{aligned} \tag{3}$$

Suppose a and b are positive. **Without solving the differential equation**, show that this system of differential equation preserves non-negativity: If $y_1(0) \geq 0$ and $y_2(0) \geq 0$, then the solution $\mathbf{y}(t) = [y_1(t) \;\; y_2(t)]^T$ satisfies $y_1(t) \geq 0$ and $y_2(t) \geq 0$ for all $t \geq 0$.

$\dot{y}_2 = -\dot{y}_1$

$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

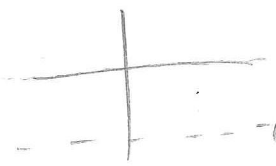
$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -ay_1 + by_2 \\ ay_1 - by_2 \end{bmatrix} = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\dot{z} = Az$$

$$\lambda^2 - T\lambda + D = \lambda^2 + (a+b)\lambda + (\cancel{ab - ab})^{0}$$

$$= \lambda(\lambda + a + b) = 0$$

$$\lambda = 0, \; -(a+b) < 0$$



Since the 2 equilibrium points are at 0 and $-(a+b)$, any IVP with initial values ≥ 0 will not cross into the negative region. Thus, if y_{00} and $y_{10} \geq 0$, $y_1(t)$ and $y_2(t) \geq 0$ for all t.