Math 33B
Differential Equations

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to **write your name**, **section** and **UID** in the space below. Additionally, write your **UID** at the **top-right corner of every page**.

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Section: 2D

Number of extra pages:

Question	Points	Score
1	10	
2	10	
3	8	
4	12	
Total:	40	

Problem 5 (in the last page) is a bonus problem for 5*pts*.

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Problem 1.

Find the general solution of the following differential equations

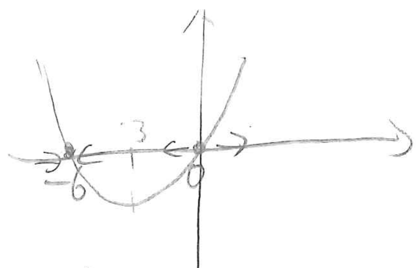
(a) [6pts.] Consider the autonomous differential equation

$$\frac{dy}{dx} = 6y + y^2,$$

Find the equilibrium points of the differential equation and characterize each equilibrium point as asymptotically stable or unstable. Justify your characterization of the equilibria using the first-derivative test.

$6y+y^2 = y(6+y)=0$

equalibrium points: $y_1=0$ $y_2=-6$.



Stable at $y=-6$

unstable at $y=0$.

$\frac{d(6y+y^2)}{dy} = 2y+6.$

when $y=-6$.

It's negative so stable.

when $y=0$.

It's positive So unstable.

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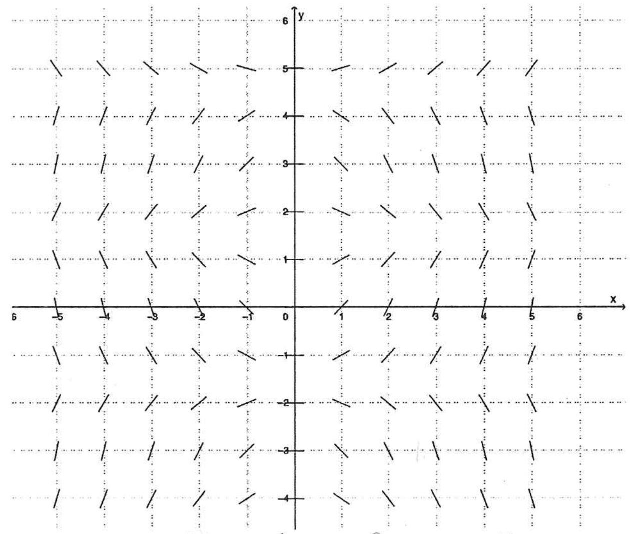
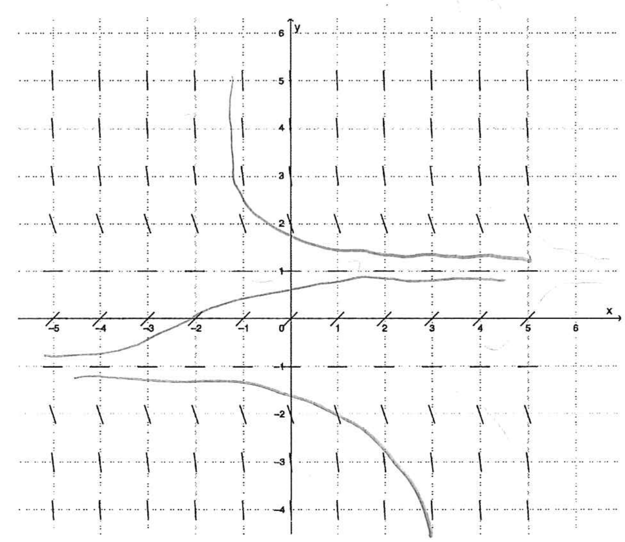
(b) [2pts.] Can the following be a direction field for a **first order autonomous differential equation**? If not, explain why.



No, it can't be. Because there's no asymptotical horizontal lines. The slope varies when x changes. If it's autonomous, the slopes is only dependent on y.

(c) [2pts.] Can the following figure be a direction field for a **autonomous first order linear differential equation**? If not, explain why.



Yes.

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Problem 2. *10pts.*

Find the general solution to the differential equation.

$$x''(t) + 2x'(t) + 2x(t) = e^{5t}. \tag{1}$$

$\lambda^2 + 2\lambda + 2 = 0$

$(\lambda+1)^2 = -1$

$\lambda = -1 \pm i$

$X_g(t) = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$

IF $X_p(t) = Ae^{5t}$

$X_p''(t) + 2X_p'(t) + 2X_p(t) = 25Ae^{5t} + 10Ae^{5t} + 2Ae^{5t} = e^{5t}$

$37A = 1$

$A = \frac{1}{37}$

$X_p(t) = \frac{1}{37} e^{5t}$

$\therefore X(t) = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t) + \frac{1}{37} e^{5t}$

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Problem 3. *8pts.*

Consider the equation

$$my''(t) + \mu y'(t) + ky(t) = 0 : \tag{2}$$

Suppose $m = 4$ and $k = 10$. For what conditions of (non-negative) μ is the system i) Underdamped ii) Overdamped iii) Critically damped. You **do not** need to find the general solution of the the differential equation.

$m\lambda^2 + \mu\lambda + k = 0 \quad \Delta = \mu^2 - 4mk = \mu^2 - 160$

(i) $\mu^2 - 4mk < 0 \quad \mu^2 < 160 \quad \mu < 4\sqrt{10}$

(ii) $\mu^2 - 4mk > 0 \quad \mu > 4\sqrt{10}$

(iii) $\mu^2 - 4mk = 0 \quad \mu = 4\sqrt{10}$

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Problem 4.

(a) [3pts.] Which one of these are possible pairs of solutions to a planar linear system:

i) $\begin{bmatrix} 5e^{3t} \\ 10e^{3t} \end{bmatrix}, \begin{bmatrix} -2e^{4t} \\ -4e^{4t} \end{bmatrix}$ ii) $\begin{bmatrix} e^{-6t} \\ e^{-6t} \end{bmatrix}, \begin{bmatrix} -e^{6t} \\ e^{6t} \end{bmatrix}$ iii) $\begin{bmatrix} 9e^{it} \\ 9ie^{it} \end{bmatrix}, \begin{bmatrix} e^{-it} \\ ie^{-it} \end{bmatrix}$

For each pair that is not possible, provide one or two lines of explanation stating why it is not a possible pair. For the pairs that can be solutions, you **do not need to** justify your answer.

(ii) Can be solution

(i) and (iii) can't be solutions because the pairs are linearly dependent.

$$\det\begin{bmatrix} 5e^{3t} & -2e^{4t} \\ 10e^{3t} & -4e^{4t} \end{bmatrix} = 0.$$

$$\det\begin{bmatrix} 9e^{it} & e^{-it} \\ 9ie^{it} & ie^{-it} \end{bmatrix} = 0.$$

(b) [7pts.] Find the solution to the following IVP

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix}\mathbf{y}; \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\det(A-\lambda I) = \lambda^2 - 2\lambda + 10 = 0.$$

$$(\lambda-1)^2 = -9$$

$$\lambda_1 = 1+3i \quad \lambda_2 = 1-3i.$$

$$(A-\lambda_1 I) = \begin{bmatrix} -1-3i & 1 \\ -10 & 1-3i \end{bmatrix}$$

$$\begin{bmatrix} -1-3i & 1 \\ -10 & 1-3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

$$(-1-3i)a + b = 0.$$

$$b = (1+3i)a.$$

take $a = 1$, $b = 1+3i$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$\therefore z(t) = e^{\lambda_1 t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = e^{t}(\cos(3t) + i\sin(3t))\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= e^{t}\left(\cos(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) + ie^{t}\left(\sin(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$\therefore y(t) = C_1 e^{t}\left(\cos(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) + C_2 e^{t}\left(\sin(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$y(0) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore C_1 = 1, \ C_2 = 0. \quad y(t) = e^{t}\left(\cos(3t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \begin{bmatrix} e^t \cos(3t) \\ e^t \cos(3t) - 3e^t \sin(3t) \end{bmatrix} = \begin{bmatrix} e^t \cos(3t) - 3e^t \sin(3t) \\ -10 e^t \cos(3t) + 2 e^t \cos(3t) - 2 \end{bmatrix}$$

$$\begin{bmatrix} e^t 3\sin(3t) + \cos(3t) e^t \\ -e^t 3\sin(3t) + \cancel{e^t} \cos(3t) + e^t \cos(3t) \\ -3e^t \sin(3t) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix}$$

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(c) [2pts.] Find the solution to the following IVP for a second order equation **using** the solution of the linear system that you found in **part b)** of this problem.

$$x'' - 2x' + 10x = 0; \quad x(0) = 1, x'(0) = 1$$

$X'' = 2X' - 10X$

if $\vec{X} = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}$ $\qquad \vec{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\dot{\vec{X}} = \begin{bmatrix} \dot{X} \\ \ddot{X} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \begin{bmatrix} X \\ \dot{X} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \vec{X}$

$\therefore \vec{X}(t) = e^t \left(\cos(3t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(3t) \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right)$

$\therefore X(t) = e^t \cos(3t)$.

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Problem 5(Bonus problem) 5*pts*.
Consider the IVP

$$\begin{aligned} \dot{y}_1 &= -ay_1 + by_2 \\ \dot{y}_2 &= ay_1 - by_2 \\ y_1(0) &= y_{00} \quad y_2(0) = y_{10} \end{aligned} \tag{3}$$

Suppose a and b are positive. **Without solving the differential equation**, show that this system of differential equation preserves non-negativity: If $y_1(0) \geq 0$ and $y_2(0) \geq 0$, then the solution $\mathbf{y}(t) = [y_1(t) \ \ y_2(t)]^T$ satisfies $y_1(t) \geq 0$ and $y_2(t) \geq 0$ for all $t \geq 0$.

$\dot{\vec{y}}(t) = \begin{bmatrix} \dot{y}_1(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix} \vec{y}(t)$

if $y_1(0) > 0$ $y_2(0) > 0$.

first, Since $\dot{y}_1 + \dot{y}_2 = 0$.

we know $(y_1 + y_2)' = 0$

Therefore $y_1(t) + y_2(t) = y_{00} + y_{10} \geq 0$.

Proof: by Contradiction, if $y_1 < 0$, then there must exists a moment when $y_1 = 0$ and $y_1' < 0$.

When $y_1 = 0$, $y_2 = y_{00} + y_{10} - y_1 \geq 0$.

$\dot{y}_1 = -ay_1 + by_2 = by_2 \geq 0$. Impossible.

$\therefore$ y_1 can't be less than 0.

by Symmetry, y_2 can't be less than 0.