19F-MATH33B-2 Midterm 1

MAGGIE YAN

TOTAL POINTS

37 / 45

QUESTION 1

Separable Equations 10 pts

1.1 SE1 5 / 5

✓ + **2 pts** **Separated variables correctly**

✓ + **1 pts** **Integrated with respect to y correctly**

✓ + **1 pts** **Integrated with respect to x correctly**

✓ + **1 pts** **+ C**

+ **0.5 pts** Bonus: mentioned change of variables or used u-sub

+ **1.5 pts** Bonus: mention and explicitly state change of variables

+ **0 pts** No work

+ **0.5 pts** Small error when integrating with respect to y

+ **1.5 pts** Minor mistake when separating variables

+ **0.5 pts** Small error when integrating with respect to x

1.2 SE2 5 / 5

✓ + **2 pts** **Correctly separated variables**

✓ + **1 pts** **Integrated with respect to x correctly**

✓ + **1 pts** **Integrated with respect to t correctly**

✓ + **1 pts** **+C**

+ **0.5 pts** Bonus: mentioned change of variables or used u-sub

+ **1.5 pts** Bonus: mention change of variables and explicitly state formula

+ **0.5 pts** Minor mistake when integrating with respect to x

+ **1.5 pts** Minor mistake when separating variables

+ **0.5 pts** Minor mistake when integrating with respect to t

QUESTION 2

Exact Differential Equations 12 pts

2.1 EDE1 4 / 4

✓ + **4 pts** **Correct**

+ **2 pts** small error

+ **1 pts** dP/dy=dQ/dx only

+ **1 pts** Said dQ/dy=dP/dx, but then proceeded correctly

+ **0 pts** incorrect

2.2 EDE2 8 / 8

✓ - **0 pts** **Correct**

QUESTION 3

3 First Order Linear Equation 8 / 8

✓ - **0 pts** **Correct: Found general solutions, but not necessarily the solution to the IVP.**

QUESTION 4

Existence and Uniqueness 10 pts

4.1 Existence 5 / 5

✓ - **0 pts** **Correct**

4.2 Uniqueness 2 / 5

✓ - **3 pts** **Incorrect usage of the theorem**

QUESTION 5

5 Bonus Question 0 / 5

+ **5 pts** Correct.

✓ + **0 pts** **Need to start with solutions to implicit equation, then prove using multivariable chain rule that solutions to implicit equation are also solutions of the differential equation.**

+ **1 pts** Mentioned chain rule, but need additional explanation or used incorrect argument.

+ **0 pts** Not attempted.

+ **2 pts** Correct direction, but need more explanation. What do your computations imply?

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Math 33B
Differential Equations

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, section and UID in the space below.

Name: Maggie Yan
Student ID number: 805126123
Section: 2E
Number of extra pages:

Question	Points	Score
1	10	
2	12	
3	8	
4	10	
Total:	40	

Bonus $+1.5pts$ each time you state and use the change of variables formula **wherever appropriate.**

Problem 5 (in the last page) is a bonus problem for $5pts$.

Problem 1.

Find the general solution of the following differential equations

(a) [5pts.]

$$\frac{dy(x)}{dx} = e^{x+4y(x)},$$

$$\frac{dy}{dx} = e^{x+4y}$$

$$\frac{dy}{dx} = e^{x}e^{4y}$$

$$\frac{1}{e^{4y}}\frac{dy}{dx} = e^{x}$$

COV: $\int \frac{1}{e^{4y}}\frac{dy}{dx}\,dx = \int e^{x}dx$

$$-\frac{1}{4}e^{-4y} = e^{x} + C$$

$$e^{-4y} = -4e^{x} - 4C$$

$$-4y = \ln(-4e^{x} - 4C)$$

$$\boxed{y = -\frac{1}{4}\ln(-4e^{x} - 4C)}$$

(b) [5pts.]

$$\frac{dx(t)}{dt} = \frac{t^4 + \sin(t)}{\cos(x(t)) + 2},$$

$$\frac{dx}{dt} = \frac{t^4 + \sin t}{\cos(x) + 2}$$

$$(\cos x + 2)\frac{dx}{dt} = t^4 + \sin t$$

COV: $\int (\cos x + 2)\frac{dx}{dt}\,dt = \int (t^4 + \sin t)\,dt$

$$2x + \sin x = \frac{1}{5}t^5 - \cos t + C$$

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Problem 2.

(a) [4pts.] Consider the differential equation

$$\underbrace{x - x\big(y(x)\big)^2}_{P} + \underbrace{\big(y(x) - kx^2y(x)\big)}_{Q}\frac{dy(x)}{dx} = 0$$

Using the definition of exactness of a differential equation, find a value for the unknown constant k so that the above differential equation is exact. You **do not need to solve** the differential equation.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial}{\partial y}(x - x(y)^2) = \frac{\partial}{\partial x}(y - kx^2y)$$

$$-2xy = -2kxy$$

$$\boxed{1 = k}$$

(b) [8pts.] Check if the following differential equation is exact. If so, find the general solution of the differential equation.

$$\underbrace{12xy(x)}_{P} + \underbrace{6\big(x^2 - (y(x))^2\big)}_{Q}\frac{dy(x)}{dx} = 0$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ✓ $\Rightarrow$ exact!

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(12xy) = 12x$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(+6(x^2 - y^2)) = \frac{\partial}{\partial x}(+6x^2 - 6y^2) = +12x$$

$$\frac{\partial f}{\partial x} = P \Rightarrow f = \int 12xy\,dx = 6x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = Q \Rightarrow \frac{\partial}{\partial y}(6x^2y + g(y)) = 6x^2 - 6y^2$$

$$6x^2 + g'(y) = 6x^2 - 6y^2$$

$$g'(y) = -6y^2 \Rightarrow g(y) = -2y^3 + C$$

$$f = \boxed{6x^2y - 2y^3 = C}$$

Problem 3. *8pts.*

Let $a(t)$ and $f(t)$ be continuous functions and let y_0 be some real number. Using any method of your choice, solve the following initial value problem,

$$\frac{dy(t)}{dt} = a(t)y(t) + f(t), \quad y(0) = y_0$$

You must show all the steps required to solve the problem.

① $\frac{dy}{dt} - a(t)y(t) = f(t)$

② $M = e^{-\int a(t)dt}$

③ $e^{-\int a(t)dt}\left(\frac{dy}{dt} - a(t)y\right) = e^{-\int a(t)dt} f(t)$

$\frac{d}{dt}\left(e^{-\int a(t)dt} y\right) = e^{-\int a(t)dt} f(t)$

$e^{-\int a(t)dt} y(t) = \int e^{-\int a(t)dt} f(t)\, dt + C$

④ $y(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt} f(t)\, dt + e^{\int a(t)dt} C$

⑤ $y_0 = e^{\int a(0)dt} \int e^{-\int a(0)dt} f(0)\, dt + Ce^{\int a(0)dt}$

$C = \frac{y_0 - e^{\int a(0)dt} \int e^{-\int a(0)dt} f(0)\, dt}{e^{\int a(0)dt}} = \frac{y_0}{e^{\int a(0)dt}} - \int e^{-\int a(0)dt} f(0)\, dt$

$$\boxed{y(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt} f(t)dt + \left(\frac{y_0}{e^{\int a(0)dt}} - \int e^{-\int a(0)dt} f(0)\, dt\right) e^{\int a(t)dt}}$$

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Problem 4.

Consider the initial value problem (IVP) : $\frac{dp(t)}{dt} = (p(t) + 4t)^{\frac{1}{3}}$, $p(1) = 4$.

(a) [5pts.] Can we guarantee that this IVP has a solution? If not, can we use the existence theorem to say that no solution to this IVP exists. Justify your answer.

$f(t,p) = (p+4t)^{1/3}$

$f(1,4) = (4+4(1))^{1/3} = 8^{1/3} = 2$

by the existence theorem, we can guarantee that a solution exists because $f(t,p) = \frac{dp}{dt}$ is defined and continuous at (t_0, p_0).
And thus, we can define rectangle R around (t_0, p_0) where a solution to the IVP exists.

can't use existence theorem to conclude if conditions not met.

(b) [5pts.] Can we guarantee that a unique solution of the IVP exists? If not, can we say that there exist multiple solutions to this IVP by applying the uniqueness theorem? Justify your answer.

$f(t,p) = (p+4t)^{1/3}$

$\frac{df}{dt} = \frac{1}{3}(p+4t)^{-2/3}\left(\frac{dp}{dt}+4\right)$

at (1,4) this $\neq 0$ so both $f(t,p)$ and $\frac{df}{dt}$ are defined and continuous at (t_0, p_0).

by the uniqueness theorem, we can guarantee that there exists a unique sol.

If conditions of the uniqueness thm aren't met, results are inconclusive.

Problem 5(Bonus problem) 5*pts*.
Show that if the differential equation,

$$P(y, t(y)) + Q(y, t(y))\frac{dt(y)}{dy} = 0$$

is exact, then there exists a function $R(p, q)$ such that the general solution $t(y)$ of the differential equation is given by

$$R(y, t(y)) = C$$

given $P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0$ is exact,

we know $\frac{\partial P}{\partial t} = \frac{\partial Q}{\partial y}$.

$$\frac{\partial f}{\partial y} = P \Rightarrow f = \int P(y,t(y))\,dy + g(y)$$

$$\frac{\partial f}{\partial t} = Q \Rightarrow \frac{d}{dt}\left(\int P(y,t(y))\,dy + g(y)\right) = Q(y,t(y))$$

$$g'(y) = Q(y,t(y)) - \frac{d}{dt}\left(\int P(y,t(y))\,dy\right)$$

$$g(y) = \int Q(y,t(y))\,dt - \int P(y,t(y))\,dy + C$$

$$f = \int Q(y,t(y))\,dt + C = 0$$

$$\int Q(y,t(y))\,dt = C$$

$R(y,t(y)) = C$ is the general sol.