Math 33B
Differential Equations

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, section and UID in the space below.

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Section: 2D.
Number of extra pages:

Question	Points	Score
1	10	
2	12	
3	8	
4	10	
Total:	40	

Bonus $+1.5pts$ each time you state and use the change of variables formula **wherever appropriate.**

Problem 5 (in the last page) is a bonus problem for $5pts$.

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Problem 1.

Find the general solution of the following differential equations

(a) [5pts.]

$$\frac{dy(x)}{dx} = e^{x+4y(x)},$$

$$y' = e^{x} \cdot e^{4y(x)}$$

$$e^{-4y(x)} y' = e^{x}$$

change of variable

$$u(x) = y(x).$$

$$\int e^{-4y(x)} y'(x)\, dx = \int e^{x} dx.$$

$$d\,u(x) = y'(x)\, dx$$

$$\int e^{-4u(x)} du(x) = \int e^{x} dx$$

$$-\frac{1}{4} e^{-4u(x)} = e^{x} + c.$$

$$-\frac{1}{4} e^{-4y(x)} = e^{x} + c$$

$$e^{-4y(x)} = -4e^{x} + c$$

$$y(x) = -\frac{1}{4}\ln(-4e^{x}+c).$$

(b) [5pts.]

$$\frac{dx(t)}{dt} = \frac{t^4 + \sin(t)}{\cos(x(t)) + 2},$$

$$(\cos(x(t)) + 2)\, x' = t^4 + \sin t$$

$$\int (\cos(x(t)) + 2)\, x' dt = \int (t^4 + \sin t)\, dt$$

change of variable.

$$u(t) = x(t)$$

$$d\,u(t) = x'(t)\, dt.$$

$$\therefore \int \cos(u(t)) + 2 \; du = \int (t^4 + \sin t)\, dt$$

$$\sin(u(t)) + 2u(t) = \frac{1}{5}t^5 - \cos t + c.$$

$$\sin(x(t)) + 2x(t) = \frac{1}{5}t^5 - \cos t + c$$

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Problem 2.

(a) [4pts.] Consider the differential equation

$$x - x\big(y(x)\big)^2 + \big(y(x) - kx^2y(x)\big)\frac{dy(x)}{dx} = 0$$

Using the definition of exactness of a differential equation, find a value for the unknown constant k so that the above differential equation is exact. You **do not need to solve** the differential equation.

$P(x,y) = x - x\,y(x)^2$

$Q(x,y) = y(x) - kx^2y(x)$

$\frac{\partial P}{\partial y} = -2xy(x).$

$\frac{\partial Q}{\partial x} = -2kx\,y(x) = -2xy(x) = \frac{\partial P}{\partial y}$

$-2k = -2$

$k = 1.$

(b) [8pts.] Check if the following differential equation is exact. If so, find the general solution of the differential equation.

$$12xy(x) + 6\big(x^2 - (y(x))^2\big)\frac{dy(x)}{dx} = 0$$

$P = 12xy(x)$

$Q = 6x^2 - 6y(x)^2$

$\frac{\partial P}{\partial y} = 12x$

$\frac{\partial Q}{\partial x} = 12x = \frac{\partial P}{\partial y}$

exact.

$F(x,y) = 6x^2y(x) - 2y(x)^3 = C.$

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Problem 3. *8pts.*

Let $a(t)$ and $f(t)$ be continuous functions and let y_0 be some real number. Using any method of your choice, solve the following initial value problem,

$$\frac{dy(t)}{dt} = a(t)y(t) + f(t), \quad y(0) = y_0$$

You must show all the steps required to solve the problem.

$$\frac{dy(t)}{dt} - a(t)y(t) = F(t)$$

$$e^{-\int a(t)dt}\left(\frac{dy(t)}{dt} - a(t)y(t)\right) = e^{-\int a(t)dt} f(t)$$

$$\int \frac{d}{dt}\left(e^{-\int a(t)dt} y(t)\right)dt = \int e^{-\int a(t)dt} f(t)\, dt$$

$$e^{-\int a(t)dt} y(t) = \int e^{-\int a(t)dt} F(t)\, dt + C.$$

$$y(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt} F(t)dt + C\cdot e^{\int a(t)dt}.$$

$$y_0 = y(0) = e^{\int a(t)dt} \int e^{-\int a(t)dt} F(t)\, dt + C\cdot e^{\int a(t)dt}\Big|_{t=0}$$

$$C = \frac{y_0 - e^{\int a(t)dt}\int e^{-\int a(t)dt} F(t)\, dt}{e^{\int a(t)dt}}\Bigg|_{t=0.}$$

$$= y_0 e^{-\int a(t)dt} - \int e^{-\int a(t)dt} f(t)dt\,\Big|_{t=0.}$$

$$y(t) = e^{\int a(t)dt}\int e^{-\int a(t)dt} F(t)dt + \left[y_0 e^{-\int a(t)dt} - \int e^{-\int a(t)dt} f(t)dt\right]_{t=0} \cdot e^{\int a(t)dt}$$

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Problem 4.

Consider the initial value problem (IVP) : $\frac{dp(t)}{dt} = (p(t) + 4t)^{\frac{1}{3}}$, $p(1) = 4$.

(a) [5pts.] Can we guarantee that this IVP has a solution? If not, can we use the existence theorem to say that no solution to this IVP exists. Justify your answer.

$f(t, P(t)) = (P(t)+4t)^{\frac{1}{3}}$

take a rectange to be $R = (0,2) \times (0,8)$.

$(1,4) \in R$.

and $f(t, P(t))$ is continuous on R.

$\therefore$ A solution Exists in R according to existence Theorem.

(b) [5pts.] Can we guarantee that a unique solution of the IVP exists? If not, can we say that there exist multiple solutions to this IVP by applying the uniqueness theorem? Justify your answer.

$$\frac{\partial f(t, P(t))}{\partial P(t)} = \frac{1}{3} \cdot (P(t)+4t)^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{(P(t)+4t)^{\frac{2}{3}}}$$

take $R = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, 8)$.

$(1,4) \in R$.

$f(t, P(t))$ is continuous on R.

$\frac{\partial f(t, P(t))}{\partial P(t)}$ is continuous on R.

$\therefore$ There exists a unique solution according to uniqueness Theorem.

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Problem 5(Bonus problem) 5*pts.*
Show that if the differential equation,

$$P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0$$

is exact, then there exists a function $R(p,q)$ such that the general solution $t(y)$ of the differential equation is given by

$$R(y,t(y)) = C$$

If exact, $\frac{\partial P(y,t(y))}{\partial t(y)} = \frac{\partial Q(y,t(y))}{\partial y}$

$\exists R(y,t(y))$: $\frac{\partial R}{\partial y} = P$ $\frac{\partial R}{\partial t(y)} = Q$

s.t. $R(y,t(y)) = C$

$\frac{dR}{dy} = 0.$

mult. var. chain rule.

$$\frac{\partial R}{\partial y}\frac{dy}{dy} + \frac{\partial R}{\partial t(y)}\frac{d(t(y))}{dy} = \frac{\partial R}{\partial y} + \frac{\partial R}{\partial t(y)}\frac{d(t(y))}{dy} = 0$$

$\therefore P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0.$