19F-MATH33B-2 Midterm 1

TOTAL POINTS

40 / 45

QUESTION 1

Separable Equations 10 pts

1.1 SE1 5.5 / 5

✓ **+ 2 pts Separated variables correctly**

✓ **+ 1 pts Integrated with respect to y correctly**

✓ **+ 1 pts Integrated with respect to x correctly**

✓ **+ 1 pts + C**

✓ **+ 0.5 pts Bonus: mentioned change of variables or used u-sub**

+ **1.5 pts** Bonus: mention and explicitly state change of variables

+ **0 pts** No work

+ **0.5 pts** Small error when integrating with respect to y

+ **1.5 pts** Minor mistake when separating variables

+ **0.5 pts** Small error when integrating with respect to x

1.2 SE2 5.5 / 5

✓ **+ 2 pts Correctly separated variables**

✓ **+ 1 pts Integrated with respect to x correctly**

✓ **+ 1 pts Integrated with respect to t correctly**

✓ **+ 1 pts +C**

✓ **+ 0.5 pts Bonus: mentioned change of variables or used u-sub**

+ **1.5 pts** Bonus: mention change of variables and explicitly state formula

+ **0.5 pts** Minor mistake when integrating with respect to x

+ **1.5 pts** Minor mistake when separating variables

+ **0.5 pts** Minor mistake when integrating with respect to t

QUESTION 2

Exact Differential Equations 12 pts

2.1 EDE1 4 / 4

✓ **+ 4 pts Correct**

+ **2 pts** small error

+ **1 pts** dP/dy=dQ/dx only

+ **1 pts** Said dQ/dy=dP/dx, but then proceeded correctly

+ **0 pts** incorrect

2.2 EDE2 7 / 8

✓ **- 1 pts Small error**

QUESTION 3

3 First Order Linear Equation 8 / 8

✓ **- 0 pts Correct: Found general solutions, but not necessarily the solution to the IVP.**

QUESTION 4

Existence and Uniqueness 10 pts

4.1 Existence 5 / 5

✓ **- 0 pts Correct**

4.2 Uniqueness 5 / 5

✓ **- 0 pts Correct**

QUESTION 5

5 Bonus Question 0 / 5

+ **5 pts** Correct.

✓ **+ 0 pts Need to start with solutions to implicit equation, then prove using multivariable chain rule that solutions to implicit equation are also solutions of the differential equation.**

+ **1 pts** Mentioned chain rule, but need additional explanation or used incorrect argument.

+ **0 pts** Not attempted.

+ **2 pts** Correct direction, but need more explanation. What do your computations imply?

Math 33B
Differential Equations

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed. For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper. Do not forget to write your name, section and UID in the space below.

Name:
Studer ____________
Sectior ____________
Numbe ____________
____

Question	Points	Score
1	10	
2	12	
3	8	
4	10	
Total:	40	

Bonus $+1.5pts$ each time you state and use the change of variables formula **wherever appropriate.**

Problem 5 (in the last page) is a bonus problem for $5pts$.

Problem 1.
Find the general solution of the following differential equations

(a) [5pts.]

$$\frac{dy(x)}{dx} = e^{x+4y(x)},$$

$$\frac{dy(x)}{dx} = e^{x} e^{4y(x)}$$

$$e^{-4y(x)} \frac{dy(x)}{dx} = e^{x}$$

change of variables $\Rightarrow$ $\int e^{-4y(x)} \frac{dy(x)}{dx}\, dx = \int e^{x}\, dx$

$$-\frac{1}{4} e^{-4y(x)} = e^{x} + C \checkmark$$

$$\ln\left(e^{-4y(x)}\right) = -4(e^{x}+C))$$

$$-4y(x) = \ln\left(-4(e^{x}+C)\right)$$

$$\boxed{y(x) = \frac{\ln(-4(e^{x}+C))}{-4}}$$ → does not exist if $-4(e^{x}+C) \leq 0$

(b) [5pts.]

$$\frac{dx(t)}{dt} = \frac{t^4 + \sin(t)}{\cos(x(t)) + 2},$$

$$(\cos(x(t)) + 2) \frac{dx(t)}{dt} = t^4 + \sin(t)$$

change of variables $\Rightarrow$ $\int (\cos(x(t)) + 2) \frac{dx(t)}{dt}\, dt = \int t^4 + \sin(t)\, dt$

$$\boxed{\sin(x(t)) + 2x(t) = \frac{1}{5}t^5 - \cos(t) + C}$$

Problem 2.

(a) [4pts.] Consider the differential equation

$$\underbrace{x - x\big(y(x)\big)^2}_{P} + \underbrace{\big(y(x) - kx^2 y(x)\big)}_{Q}\frac{dy(x)}{dx} = 0$$

Using the definition of exactness of a differential equation, find a value for the unknown constant k so that the above differential equation is exact. You **do not need to solve** the differential equation.

For the equation to be exact, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = -2xy(x)$ $\quad \frac{\partial Q}{\partial x} = -2Kxy(x)$

$-2xy(x) = -2Kxy(x)$

$\boxed{1 = K}$

(b) [8pts.] Check if the following differential equation is exact. If so, find the general solution of the differential equation.

$$\underbrace{12xy(x)}_{P} + \underbrace{6\big(x^2 - (y(x))^2\big)}_{Q}\frac{dy(x)}{dx} = 0$$

$\frac{\partial P}{\partial y} = 12X$ $\quad \frac{\partial Q}{\partial x} = 12X$ $\quad$ Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the equation is exact.

There is a function F such that $F = \int P\,dx = \int 12x\,dx = 6x^2 + \phi(y)$

$\frac{\partial F}{\partial y} = \phi'(y) = Q = 6x^2 - 6(y(x))^2$

so $\phi(y) = \int 6x^2 - 6(y(x))^2\,dy = 6x^2y(x) - 2(y(x))^3 + C_1$

so $F = 6x^2 + 6x^2y(x) - 2(y(x))^3 + C_1 = 0$

$$\boxed{F = 6x^2 + 6x^2(y(x)) - 2(y(x))^3 = C_2}$$

Problem 3. *8pts.*
Let $a(t)$ and $f(t)$ be continuous functions and let y_0 be some real number. Using any method of your choice, solve the following initial value problem,

$$\frac{dy(t)}{dt} = a(t)y(t) + f(t), \quad y(0) = y_0$$

You must show all the steps required to solve the problem.

$$\frac{dy(t)}{dt} - a(t)y(t) = f(t)$$

$$e^{-\int a(t)dt}\left[\frac{dy(t)}{dt} - a(t)y(t) = f(t)\right]$$

$$\frac{d}{dt}\left[e^{-\int a(t)dt} y(t)\right] = e^{-\int a(t)dt} f(t)$$

$$\int \frac{d}{dt}\left[e^{-\int a(t)dt} y(t)\right] dt = \int e^{-\int a(t)dt} f(t)\, dt$$

$$e^{-\int a(t)dt} y(t) = \int e^{-\int a(t)dt} f(t)\, dt + C$$

$$y(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt} f(t)\, dt + Ce^{\int a(t)dt}$$

$$y_0 = e^{\int a(0)dt} \int e^{-\int a(0)dt} f(0)\, dt + Ce^{\int a(0)dt}$$

$$\frac{y_0}{e^{\int a(0)dt}} - \int e^{-\int a(0)dt} f(0)\, dt = C$$

so, $$\boxed{y(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt} f(t)dt + \left(\frac{y_0}{e^{\int a(0)dt}} - \int e^{-\int a(0)dt} f(0)\, dt\right) e^{\int a(0)dt}}$$

Problem 4.

Consider the initial value problem (IVP) : $\frac{dp(t)}{dt} = (p(t) + 4t)^{\frac{1}{3}}$, $p(1) = 4$.

(a) [5pts.] Can we guarantee that this IVP has a solution? If not, can we use the existence theorem to say that no solution to this IVP exists. Justify your answer.

Let $f(x_1, x_2) = (x_2 + 4x_1)^{\frac{1}{3}}$

Since $f(x_1, x_2)$ is continuous everywhere, including at $(x_1, x_2) = (1, 4)$, then by the existence theorem, we can guarantee that this IVP has a solution.

(b) [5pts.] Can we guarantee that a unique solution of the IVP exists? If not, can we say that there exist multiple solutions to this IVP by applying the uniqueness theorem? Justify your answer.

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{1}{3}(x_2 + 4x_1)^{-\frac{2}{3}} = \frac{1}{3(x_2+4x_1)^{2/3}}$$

This is not continuous when $x_2 + 4x_1 = 0$, but it is continuous for the given initial condition. Since both $(x_2+4x_1)^{\frac{1}{3}}$ and $\frac{1}{3(x_2+4x_1)^{2/3}}$ are continuous at $(x_1, x_2) = (1, 4)$, by the uniqueness theorem, we can guarantee a unique solution of the IVP exists.

Problem 5(Bonus problem) 5*pts.*
Show that if the differential equation,

$$P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0$$

is exact, then there exists a function $R(p,q)$ such that the general solution $t(y)$ of the differential equation is given by

$$R(y,t(y)) = C$$

if the equation is exact, then $\dfrac{\partial P(y,t(y))}{\partial t(y)} = \dfrac{\partial Q(y,t(y))}{\partial y}$

This tells us that there must be a function R such that its gradient is ~~$\left(\frac{\partial P(y,t(y))}{\partial t(y)}, \frac{\partial Q(y,t(y))}{\partial y}\right)$ which means~~

~~$R = \int P\,dy + \int Q\,dt(y)$~~

$\dfrac{\partial R}{\partial y} = P(y,t(y))\ \dfrac{\partial Q(y,t(y))}{dy}$ $\quad$ $\dfrac{\partial R}{\partial t(y)} = Q(y,t(y))\ \dfrac{dP(y,t(y))}{dt(y)}$