Problem 1.

Find the general solution of the following differential equations

(a) [5pts.]

$$\frac{dy(x)}{dx} = e^{x+4y(x)},$$

$$\frac{dy(x)}{dx} = e^{x}e^{4y(x)}$$

$$\Rightarrow \int e^{-4y(x)}\,dy(x) = \int e^{x}\,dx$$

$$\Rightarrow -\frac{1}{4}e^{-4y(x)} = e^{x} + c$$

$$\Rightarrow e^{-4y(x)} = -4e^{x} + c$$

$$\Rightarrow -4y(x) = \ln(-4e^{x}) + \ln(c)$$

$$\Rightarrow y(x) = -\frac{1}{4}(\ln(-4e^{x}) + \ln(c))$$ is the general explicit solution for an arbitrary constant c

(b) [5pts.]

$$\frac{dx(t)}{dt} = \frac{t^4 + \sin(t)}{\cos(x(t)) + 2},$$

$$\int (\cos(x(t)) + 2)\,dx(t) = \int (t^4 + \sin(t))\,dt$$

$$\Rightarrow \sin(x(t)) + 2x(t) = \frac{1}{5}t^5 - \cos(t) + C$$ is the general implicit solution for an arbitrary constant C

Problem 2.

(a) [4pts.] Consider the differential equation

$$x - x\big(y(x)\big)^2 + \big(y(x) - kx^2y(x)\big)\frac{dy(x)}{dx} = 0$$

Using the definition of exactness of a differential equation, find a value for the unknown constant k so that the above differential equation is exact. You **do not need to solve** the differential equation.

Let $M = x - x(y(x))^2$, $N = y(x) - kx^2y(x)$

Want $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial y} = -2y(x)x$, $\frac{\partial N}{\partial x} = -2xky(x)$

For $-2y(x)x = -2xky(x)$

$\Longrightarrow k = 1$

(b) [8pts.] Check if the following differential equation is exact. If so, find the general solution of the differential equation.

$$12xy(x) + 6\big(x^2 - (y(x))^2\big)\frac{dy(x)}{dx} = 0$$

Let $M = 12xy(x)$, $N = 6x^2 - 6(y(x))^2$

Since $\frac{\partial M}{\partial y} = 12x = \frac{\partial N}{\partial x}$, the differential equation is exact.

Want f such that $f = \int M\,dx + g(y(x))$,

$f = \int 12xy(x)\,dx + g(y(x))$

$= 6x^2y(x) + g(y(x))$

Want $\frac{\partial f}{\partial y} = N$, and find $g(y(x))$.

$\frac{\partial f}{\partial y} = 6x^2 + g'(y(x))$ so $6x^2 + g'(y(x)) = 6x^2 - 6(y(x))^2$

$\Longrightarrow g'(y(x)) = -6(y(x))^2$

$\Longrightarrow \int g'(y(x))\,dy(x) = \int -6(y(x))^2\,dy(x)$

$\Longrightarrow g(y(x)) = -2(y(x))^3$

Plugging $g(y(x))$ into f,

$f(x,y) = 6x^2y(x) - 2(y(x))^3$ and setting $f(x,y) = C$

$\Longrightarrow 6x^2y(x) - 2(y(x))^3 = C$ is the general solution for an arbitrary constant C

Problem 3. *8pts.*

Let $a(t)$ and $f(t)$ be continuous functions and let y_0 be some real number. Using any method of your choice, solve the following initial value problem,

$$\frac{dy(t)}{dt} = a(t)y(t) + f(t), \quad y(0) = y_0$$

You must show all the steps required to solve the problem.

$$\frac{dy(t)}{dt} = a(t)y(t) + f(t)$$

$$\Rightarrow \frac{dy(t)}{dt} - a(t)y(t) = f(t)$$ ① set in standard form

$$\Rightarrow e^{-\int a(t)dt}\left(\frac{dy(t)}{dt} - a(t)y(t)\right) = e^{-\int a(t)dt} f(t)$$ ② Apply integrating factor $\mu = e^{-\int a(t)dt}$

$$\Rightarrow e^{-\int a(t)dt}\frac{dy(t)}{dt} - e^{-\int a(t)dt} a(t)y(t) = e^{-\int a(t)dt} f(t)$$

$$\Rightarrow e^{-\int a(t)dt}\frac{dy(t)}{dt} + \left(e^{-\int a(t)}\right)' y(t) = e^{-\int a(t)dt} f(t)$$

$$\Rightarrow \left(e^{-\int a(t)dt} y(t)\right)' = e^{-\int a(t)dt} f(t)$$ ③ Integrate and solve

$$\Rightarrow e^{-\int a(t)dt} y(t) = \int e^{-\int a(t)dt} f(t)\,dt + C$$

$$\Rightarrow y(t) = e^{\int a(t)dt}\int e^{-\int a(t)dt} f(t)\,dt + Ce^{\int a(t)dt}$$

④ Solving for $y(0) = y_0$

$$y(0) = e^{\int a(0)dt}\int e^{-\int a(0)dt} f(0)\,dt + Ce^{\int a(0)dt} = y_0$$

$$\Rightarrow Ce^{\int a(0)dt} = y_0 - e^{\int a(0)dt}\int e^{-\int a(0)dt} f(0)\,dt$$

$$\Rightarrow C = y_0 e^{-\int a(0)dt} - \int e^{-\int a(0)dt} f(0)\,dt$$

⑤ Plugging C into $y(t)$,

$$y(t) = e^{\int a(0)dt}\int e^{-\int a(0)dt} f(0)\,dt + y_0 - e^{\int a(0)dt}\int e^{-\int a(0)dt} f(0)\,dt$$

is the solution to the IVP.

Problem 4.

Consider the initial value problem (IVP) : $\frac{dp(t)}{dt} = (p(t) + 4t)^{\frac{1}{3}}$, $p(1) = 4$.

(a) [5pts.] Can we guarantee that this IVP has a solution? If not, can we use the existence theorem to say that no solution to this IVP exists. Justify your answer.

Let $f(t, p(t)) = (p(t) + 4t)^{\frac{1}{3}}$

at $f(1,4) = (4+4)^{\frac{1}{3}} = 8^{\frac{1}{3}} = 2$

Since $f(t, p(t))$ is continuous and exists in the rectangle R such that $(t_0, p_0) = (1,4) \in R$, we can say that this IVP has a solution by the existence theorem.

(b) [5pts.] Can we guarantee that a unique solution of the IVP exists? If not, can we say that there exist multiple solutions to this IVP by applying the uniqueness theorem? Justify your answer.

Let $f(t, p(t)) = (p(t) + 4t)^{\frac{1}{3}}$ and $\frac{\partial f(t, p(t))}{\partial p(t)} = \frac{1}{3}(p(t) + 4t)^{-\frac{2}{3}}$

at $f(1,4) = (4+4)^{\frac{1}{3}} = 8^{\frac{1}{3}} = 2$ and $\frac{\partial f}{\partial p} = \frac{1}{3}(4+4)^{-\frac{2}{3}} = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$

Since $f(t, p(t))$ and $\frac{\partial f(t, p(t))}{\partial p(t)}$ are continuous and exist in the rectangle R such that $(t_0, p_0) = (1,4) \in R$, we can say that a unique solution of the IVP exists by the uniqueness theorem.

Problem 5(Bonus problem) 5*pts*.
Show that if the differential equation,

$$P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0$$

is exact, then there exists a function $R(p,q)$ such that the general solution $t(y)$ of the differential equation is given by

$$R(y,t(y)) = C$$

If $\frac{\partial P}{\partial t} = \frac{\partial Q}{\partial y}$, then there exists $R(p,q)$ such that

$\frac{\partial R}{\partial y} = P$, $\frac{\partial R}{\partial t} = Q$