33B Final

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$\boxed{Q1}$ $y' - \sin t \, y^2 = 0 \quad y(0) = y_0$

Seperable ODE

a) $\frac{dy}{dt} = \sin t \cdot y^2$

$\frac{1}{y^2} dy = \sin t \, dt \Rightarrow \frac{1}{y^2} dy - \sin t \, dt = 0$

$\int \frac{1}{y^2} dy - \int \sin t \, dt = C$

$-\frac{1}{y} - (-\cos t) = C$

$-\frac{1}{y} = C - \cos t$

$\boxed{y(0) = y_0 \qquad y_0 = -\frac{1}{C-1} \qquad C = -\frac{1}{y_0} + 1}$

general solution $y = -\frac{1}{C - \cos t}$

Particular Solution $y = -\frac{1}{-\frac{1}{y_0} + 1 - \cos t}$

b) $y(t) = y_0 \qquad y_0 = \cancel{\frac{1}{C - \cos t}}$

~~$C - \cos t$ can not be zero for any t~~

$-\frac{1}{y_0} + 1 - \cos t \neq 0$

$\cos t$ ranges from $[-1, 1]$

$\frac{y_0 - 1}{y_0} - \cos t \neq 0$ thus $\left| \frac{y_0 - 1}{y_0} \right| > 1$

$\rightarrow$

$\boxed{Q1}$

(b.) cont... we find that $\cancel{y_0}\ y_0 < \frac{1}{2}$ exceeds range of cos t.

$y_0 \neq 0$

y_0 includes all $\mathbb{R}$ s.t. $y_0 < 1/2,\ y_0 \neq 0$

(c.) $y_0 = 1$

$$y(t) = -\frac{1}{-\frac{1}{1} + 1 - \cos t} = \frac{1}{\cos t}$$

interval must contain $(0, 1)$, largest maximal interval

$\cos t = 0$ at $\frac{\pi}{2}, -\frac{\pi}{2} \pm \pi n$

$-\frac{\pi}{2} < 0 < \frac{\pi}{2}$

IoE: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\boxed{Q2}$ $(2t^3-6t^2y+3ty^2)dt+(-2t^3+kt^2y-y^3)dy=0$

a. Both P, Q cont'ly diff'ble on $\mathbb{R}^2$

Show $Pdt+Qdy$ is closed

$$\frac{d}{dy}(2t^3-6t^2y+3ty^2)=\frac{d}{dt}(-2t^3+kt^2y-y^3)$$

$$-6t^2+6ty=-6t^2+2kty$$

$$6ty=2kty \Rightarrow 6=2k$$

$$k=3$$

b. Integrate; we know it's exact

$$F(t,y)=\int 2t^3-6t^2y+3ty^2\,dt=\frac{1}{2}t^4-2t^3y+\frac{3}{2}t^2y^2+\varphi(y)$$

$$\frac{dF}{dy}=\frac{d}{dy}\left(\frac{1}{2}t^4-2t^3y+\frac{3}{2}t^2y^2\right)=-2t^3+3t^2y+\varphi'(y)$$

$$-2t^3+3t^2y-y^3=-2t^3+3t^2y+\varphi'(y)$$

$$\varphi'(y)=-y^3 \qquad \int\varphi'(y)dy=\int -y^3dy$$

Sub back in $\qquad \varphi(y)=-\frac{1}{4}y^4+C$

$$F(t,y)=\frac{1}{2}t^4-2t^3y+\frac{3}{2}t^2y^2-\frac{1}{4}y^4+C$$

$$\frac{1}{2}t^4-2t^3y+\frac{3}{2}t^2y^2-\frac{1}{4}y^4=C$$ — implicit form

$\boxed{Q3}$ $y'(t) = \sin t\, y^2 - \sin^3 t + \cos t \quad y(0) = 1$

- use existence theorem: $y' = f(t, y)$

  $f(t,y)$ is continuous in a rectangle (all of $\mathbb{R}^2$) which contains the point $(0,1)$. Thus the IVP has a solution.

- use uniqueness theorem: already know $f(t,y)$ continuous on $\mathbb{R}^2$

  Show $\frac{\partial f}{\partial y}$ exists and continuous on $\mathbb{R}^2$

  $$\frac{\partial}{\partial y}(\sin t\, y^2 - \sin^3 t + \cos t) = 2\sin t \cdot y - \sin^3 t + \cos t$$

  continuous on $\mathbb{R}^2$

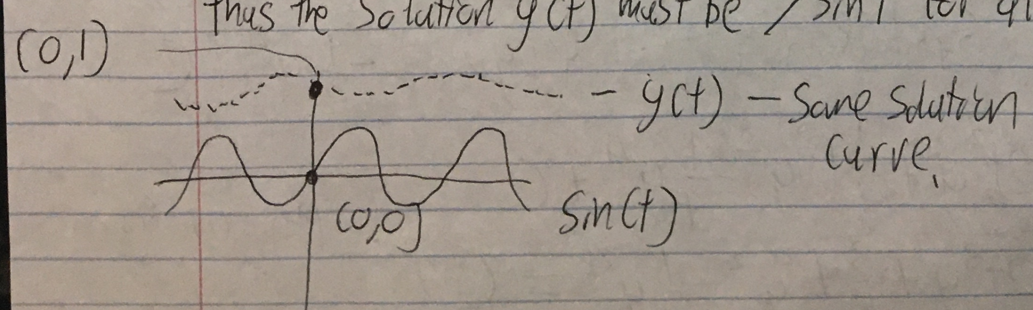
  thus we have a unique solution.

  try $y(t) = \sin t$ plug into $y'(t) = \sin t\, y^2 - \sin^3 t + \cos t$ $\quad y(0)=1$

  $$\cos t = \sin^3 t - \sin^3 t + \cos t$$
  $$\cos t = \cos t$$
  $$\sin(0) = 0 \quad (0,0)$$

  thus $y(t) = \sin t$ is a solution.

  By the uniqueness theorem, solution curves may not intersect thus the solution $y(t)$ must be $> \sin t$ for all t



(0,1) above (0,0)

$\boxed{Q4}$ $t^2y'' + ty' - 4y = 0, \quad t > 0$

(a) test $y_1(t) = t^2$

$t^2(t^2)'' + t(t^2)' - 4(t^2) = 0$, true

$2t^2 + 2t^2 - 4t^2 = 0 \Rightarrow 0 = 0$

$y_1(t) = t^2$ is a solution

(b) $y_2(t) = y_1(t)\,v(t) = t^2 v(t)$ ~ plug in

$t^2(t^2 v)'' + t(t^2 v)' - 4(t^2 v) = 0$

$2t^2 v + 4t^3 v' + t^4 v'' + 2t^2 v + t^3 v' - 4t^2 v = 0$

$t^4 v'' + 5t^3 v' = 0 \qquad$ let $w = v'$

$t^4 w' + 5t^3 w = 0 \qquad$ first order linear ODE

$w' + \frac{5}{t} w = 0 \qquad \mu(t) = e^{\int \frac{5}{t} dt} = e^{5\ln t} = t^5$

$(\mu w)' = 0 \cdot \mu$

$w t^5 = \int 0\, dt \Rightarrow w t^5 = C_1$

$w = \frac{C_1}{t^5}$

Sub in $w = v'$ $\qquad v' = \frac{C_1}{t^5}$

$v = \int \frac{C_1}{t^5} dt = -\frac{C_1}{4t^4} + C_2$ $\qquad$ ~~It appears I do not need to solve this.~~

Solved in part C $\qquad$ ~~oops~~.

$\boxed{Q4}$ (C) $t^2y''+ty'-4y=-3t-4$

$y_2(t)=vt^2=\dfrac{-C_1}{4t^2}+C_2t^2$ we can simplify C_1 $=C_1\cdot\dfrac{1}{t^2}+C_2\cdot t^2$

$y_1(t)=t^2$ $y_1+y_2=$ fundamental solution to homogenous eq

$y_h(t)=C_1\cdot\dfrac{1}{t^2}+C_2\cdot t^2$ y_1 is added into C_2

Now use variation of parameters

$y_p=V_1(t)\cdot\dfrac{1}{t^2}+V_2(t)\cdot t^2$

$$W(t)=\det\begin{pmatrix}\frac{1}{t^2} & t^2\\ -\frac{2}{t^3} & 2t\end{pmatrix}=\frac{2}{t}+\frac{2}{t}=\frac{4}{t}$$

$$y''+\frac{1}{t}y-\frac{4}{t^2}y=\frac{-3t-4}{t^2}\quad g(t)$$

$$V_1(t)=\int\frac{-y_2(t)g(t)}{W(t)}=\int\frac{-t^2\frac{-3t-4}{t^2}}{\frac{4}{t}}dt=\frac{1}{4}\int 3t^2+4t\,dt=\frac{1}{4}t^3+\frac{1}{2}t^2$$

$$V_2(t)=\int\frac{y_1(t)g(t)}{W(t)}=\int\frac{\frac{1}{t^2}\cdot\frac{-3t-4}{t^2}}{\frac{4}{t}}dt=\frac{1}{4}\int-\frac{3}{t^2}-\frac{4}{t^3}dt=\frac{+3}{4t}+\frac{1}{2t^2}$$

$$y_p=\frac{1}{4}t+\frac{1}{2}+\frac{3}{4}t+\frac{1}{2}=t+1\qquad y_p=t+1$$

$\boxed{Q5}$ $A = \begin{bmatrix} -4 & 1 \\ -1 & -4 \end{bmatrix}$

a. $x' = Ax$

$\det(A - \lambda I) = \det\begin{pmatrix} -4-\lambda & 1 \\ -1 & -4-\lambda \end{pmatrix} = (-4-\lambda)^2 + 1$

Char eq $= 16 + 8\lambda + \lambda^2 + 1 = \lambda^2 + 8\lambda + 17 = 0$

use quad formula $\dfrac{-8 \pm \sqrt{64-68}}{2} = -4 \pm i = \lambda_1, \lambda_2$

$\lambda_1 = -4 + i$

null $(A - (-4+i)I) \Rightarrow \left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right) \xRightarrow[\text{rref}]{\text{Swap rows}} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = -i x_2$

$x_2 = x_2$

$V_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ Since $\overline{\lambda_1} = \lambda_2$

$V_2 = \overline{V_1}$

$\lambda_2 = -4 - i$

$V_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ = conjugate of V_1

use real equation to solve! let $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ real

$\lambda = -4 \pm i$ $\alpha = -4, \beta = 1$ $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ imag

gen Solution

$$X(t) = c_1 e^{-4t}\left(\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + c_2 e^{-4t}\left(\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Q5 (b.) $x(0)=\begin{bmatrix}2\\1\end{bmatrix}$

$$X(t)=C_1e^{-4t}\left(\cos t\begin{pmatrix}0\\1\end{pmatrix}-\sin t\begin{pmatrix}1\\0\end{pmatrix}\right)+C_2e^{-4t}\left(\cos t\begin{pmatrix}1\\0\end{pmatrix}+\sin t\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$X(0)=C_1\cdot\begin{pmatrix}0\\1\end{pmatrix}+C_2\cdot\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}C_2\\C_1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

$C_2=2,$

$C_1=1$

$$X(t)=e^{-4t}\left(\cos t\begin{pmatrix}0\\1\end{pmatrix}-\sin t\begin{pmatrix}1\\0\end{pmatrix}\right)+2e^{-4t}\left(\cos t\begin{pmatrix}1\\0\end{pmatrix}+\sin t\begin{pmatrix}0\\1\end{pmatrix}\right)$$

Solution to IVP;

$\boxed{Q6}$ $x_1' = 4x_1 + x_2$ (1)

$x_2' = x_1 + 4x_2 + 2e^{3t}$ (2)

(a) $x_2(t) = x_1'(t) - 4x_1(t)$ plug into eq 2

$$(x_1'(t) - 4x_1(t))' = x_1(t) + 4(x_1'(t) - 4x_1(t)) + 2e^{3t}$$

$$x_1'' - 4x_1' = x_1 + 4x_1' - 16x_1 + 2e^{3t}$$

$$x_1'' - 8x_1' + 15x_1 = 2e^{3t}$$ inhomogenous 2nd order

(b.) First find y_p using undetermined coefficients

trial Solution $y_p(t) = Ae^{3t}$ — note this solves the corresponding homogenous equation. Multiply trial solution by t

$$Ate^{3t}$$

$$(Ate^{3t})'' - 8(Ate^{3t})' + 15(Ate^{3t}) = 2e^{3t}$$

$$9Ate^{3t} + 3Ae^{3t} + 3Ae^{3t} - 8Ae^{3t} - 24Ate^{3t} + 15Ate^{3t} = 2e^{3t}$$

cancel terms

$$-2Ae^{3t} = 2e^{3t} \Rightarrow A = -1$$

$$y_p(t) = -te^{3t}$$

$X_p(t)$ for x_1

Q6 (b) Cont... Solve for $y_h(t)$

$x_h(t)$: $x_1'' - 8x_1' + 15x_1 = 0$ Second order homogenous eq

Char eq: $\lambda^2 - 8\lambda + 15 = 0$

$(\lambda - 3)(\lambda - 5) = 0 \quad \lambda_1 = 3, \lambda_2 = 5$

$x_h(t) = C_1 e^{3t} + C_2 e^{5t}$

$x_1(t) = x_h(t) + x_p(t)$

$x_1(t) = C_1 e^{3t} + C_2 e^{5t} - te^{3t}$

(c) Plug into eq (1) $\quad x_1' = 4x_1 + x_2$

$(C_1 e^{3t} + C_2 e^{5t} - te^{3t})' = 4(C_1 e^{3t} + C_2 e^{5t} - te^{3t}) + x_2$

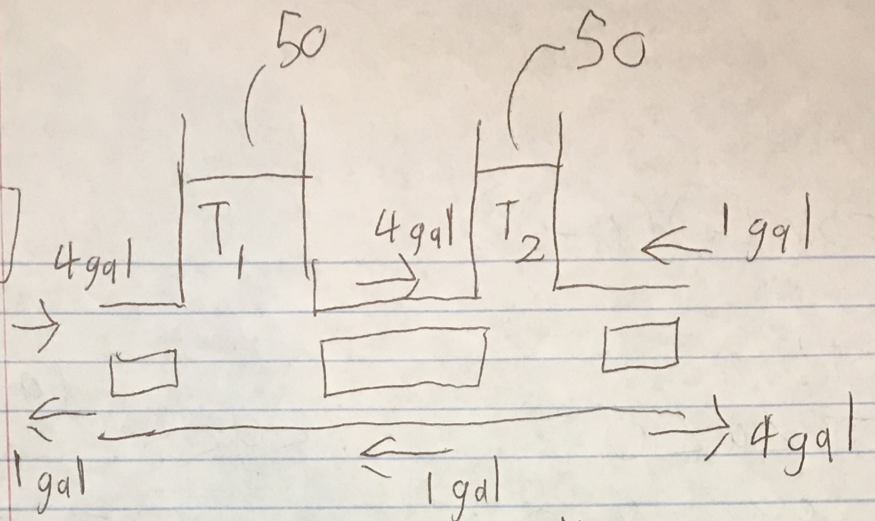
$3C_1 e^{3t} + 5C_2 e^{5t} - e^{3t} - 3te^{3t} = 4C_1 e^{3t} + 4C_2 e^{5t} - 4te^{3t} + x_2$

$-C_1 e^{3t} + C_2 e^{5t} + te^{3t} - e^{3t} = x_2(t)$

Simplify C_1

$x_2(t) = C_1 e^{3t} + C_2 e^{5t} + te^{3t} - e^{3t}$

Q7



Note: all volumes are constant; flow in = flow out

$4-4+1-1=0$

$4-4+1-1=0$

balance condition

$$x_1'(t) = -\underbrace{\frac{x_1(t)}{50}}_{\text{to outside}} - \underbrace{\frac{4x_1(t)}{50}}_{\text{to tank 2}} + \underbrace{\frac{x_2(t)}{50}}_{\text{from tank 2}}$$

flowing out — flowing in

$$x_2'(t) = -\underbrace{\frac{x_2(t)}{50}}_{\text{to tank 1}} - \underbrace{\frac{4x_2(t)}{50}}_{\text{to outside}} + \underbrace{\frac{4x_1(t)}{50}}_{\text{from tank 1}} + \underbrace{0.42}_{\text{from pipe}}$$

flowing out — flowing in

lbs/gal · 1 gal/min

$$\boxed{\frac{\text{lbs}}{\text{min}} = \frac{\text{lbs}}{\text{gal}} \cdot \frac{\text{gal}}{\text{min}}}$$

$$x_1' = \frac{-5x_1}{50} + \frac{x_2}{50}$$

$$x_2' = \frac{-5x_2}{50} + \frac{4x_1}{50} + 0.42 \quad \text{non homogenous}$$

$$x' = \begin{pmatrix} -\frac{1}{10} & \frac{1}{50} \\ \frac{2}{25} & -\frac{1}{10} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.42 \end{pmatrix}$$

where $x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

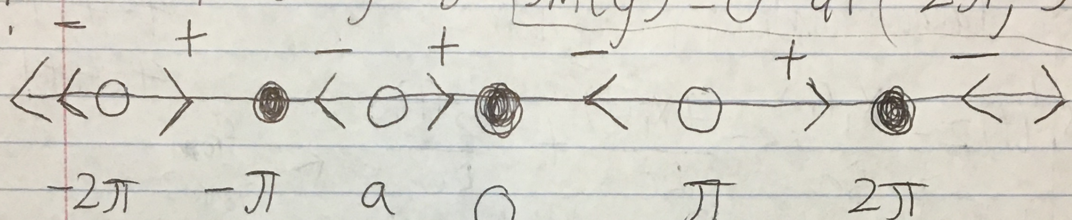
$\boxed{Q8}$ $y' = \sin(y)(a-y)$

(a.) First:

(i) $-\pi < a < 0$ $\quad -8 < y < 8$ $\quad (a-y)=0$ when $y=a$

within $-8<y<8$ $\boxed{\sin(y)=0}$ at $(-2\pi, -\pi, \pi, 2\pi, 0)$

Phase line:



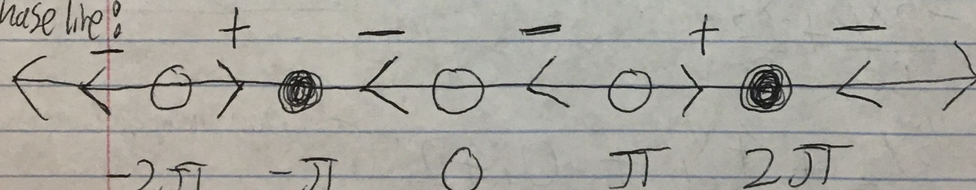
Equilibrium Points: $(-2\pi, -\pi, a, 0, \pi, 2\pi) = y$

unstable: $y = -2\pi, a, \pi$

asymptotically stable: $y = -\pi, 0, 2\pi$

(ii) $a=0$ $\quad -8<y<8$ $\quad$ Since $a=0$ we have one less equilibrium point

Phase line:



Equilibrium Points: $(-2\pi, -\pi, 0, \pi, 2\pi) = y$

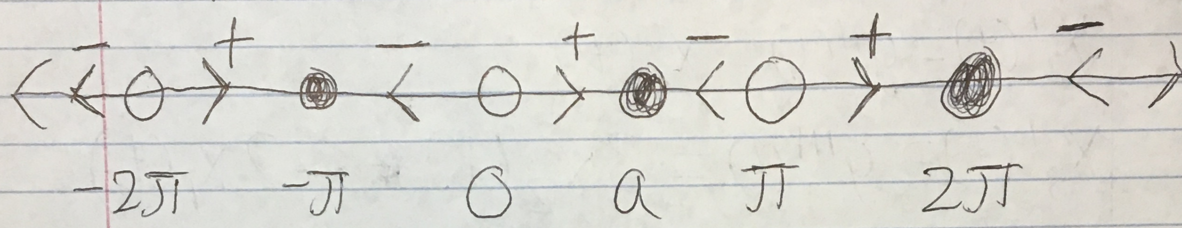
unstable: $y = -2\pi, 0, \pi$

asymptotically stable: $y = -\pi, 2\pi$

$\boxed{Q8}$ (a). Cont...

(iii) $0 < a < \pi \qquad -8 < y < 8$

Same equilibrium points as (i) $\qquad (a-y)=0$ when $y=a$



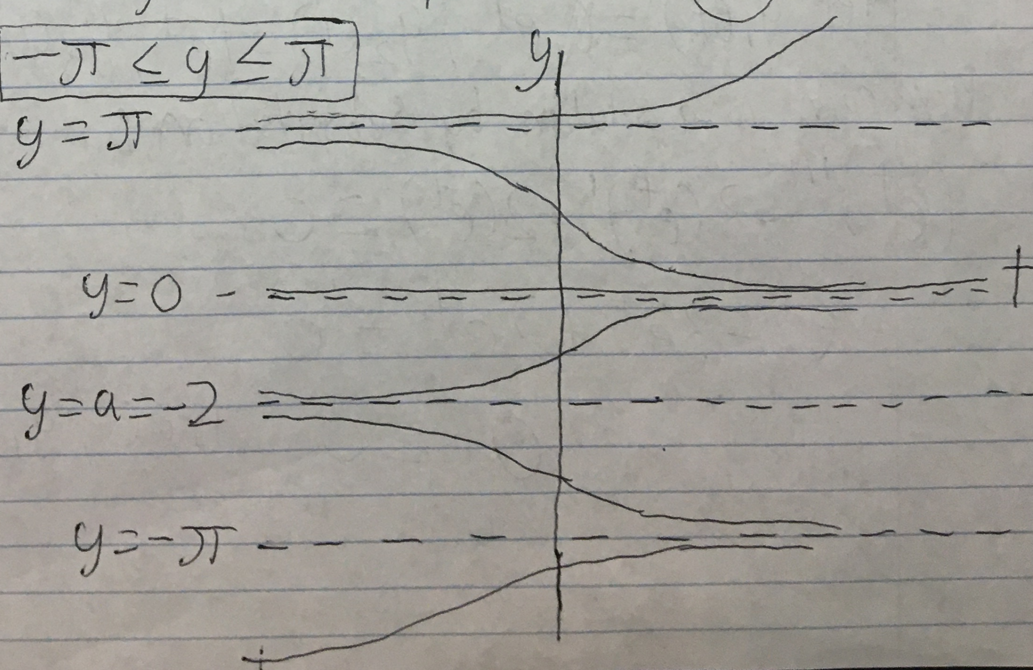
Equilibrium Points : $y = (-2\pi, -\pi, 0, a, \pi, 2\pi)$

unstable :
$y = -2\pi, 0, \pi$

asymptotically stable :
$y = -\pi, a, 2\pi$

(b.) $a = -2$, We can use phase line in (i) for guidance,

$\boxed{-\pi \le y \le \pi}$



$\boxed{Q9}$ $y''' - 3y' + 2y = 0$

a.) let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = y''(t)$

$x_1'(t) = y'(t) = x_2(t)$

$x_2'(t) = y''(t) = x_3(t)$

$x_3'(t) = y'''(t) = 3x_2(t) - 2x_1(t)$

$$x' = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}}_{A} x \qquad x = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

b.) $y(t) = C_1 e^{t} + C_2 t e^{t} + C_3 e^{-2t}$

let $y_1(t) = e^{t}$, $y_2(t) = te^{t}$, $y_3(t) = e^{-2t}$

Prove each is a solution by substitution:

$y_1(t)$:

$(e^{t})''' - 3(e^{t})' + 2(e^{t}) = 0$

$e^{t} - 3e^{t} + 2e^{t} = 0$

$0 = 0$ true, $y_1(t)$ is a solution

$\boxed{Q9}$ b. cont...

$y_2(t)$:

$$(te^t)''' - 3(te^t)' + 2(te^t) = 0$$

$$te^t + e^t + e^t + e^t - 3te^t - 3e^t + 2te^t = 0$$

$$0 = 0 \quad \text{true}, \ y_2(t) \text{ is a Solution}$$

$y_3(t)$:

$$(e^{-2t})''' - 3(e^{-2t})' + 2(e^{-2t}) = 0$$

$$-8e^{-2t} + 6e^{-2t} + 2e^{-2t} = 0$$

$$0 = 0 \quad \text{true}, \ y_3(t) \text{ is a Solution}$$

Showing $W(t) \neq 0$ for any value t implies y_1, y_2, y_3 form a fundamental Set of Solutions

$$W(t) = \det\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} = \det\begin{pmatrix} e^t & te^t & e^{-2t} \\ e^t & te^t + e^t & -2e^{-2t} \\ e^t & te^t + 2e^t & 4e^{-2t} \end{pmatrix}$$

let t be 0

$$= \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -2 \\ 1 & 2 & 4 \end{pmatrix} = (4+4) + 0 + (2-1) = 9 \neq 0$$

thus $y_1(t) = e^t$, $y_2(t) = te^t$, $y_3(t) = e^{-2t}$ form a fundamental triple

C.

$x_1(t) = y(t)$

$\boxed{Q9}$ we are given $y(t) = C_1 e^t + C_2 t e^t + C_3 e^{-2t}$

Note

$x_1(t) = C_1 e^t + C_2 t e^t + C_3 e^{-2t}$

$x_2 = x_1'$ $x_2(t) = C_1 e^t + C_2 t e^t + C_2 e^t + (-2C_3 e^{-2t})$

$x_3 = x_2'$ $x_3(t) = C_1 e^t + C_2 t e^t + 2C_2 e^t + 4C_3 e^{-2t}$

with this we form a general solution $x(t) = C_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^t \left(t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right) + C_3 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

we can further reverse engineer:

No sin/cos shows not complex

$C_1 e^{\lambda t} \Rightarrow \lambda_1 = 1$, $C_2 e^{\lambda t} \Rightarrow \lambda_2 = 1$

for E_1: we have a double eigen value

either we have eigen basis or we don't

we see we don't; our general solution has form

$C_1 e^{\lambda t} \vec{v}_1 + C_2 e^{\lambda t} (\vec{v}_2 + t\vec{v}_1)$

In no other cases do we have anything resembling this form

we see $C_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^t \left(t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$ no other eigen vector

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponds to v_1 $E_1 = \text{Span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ no eigen basis

$\boxed{Q9}$ C_1 Cont...

For E_1 we do not have a full basis and thus there is <u>no</u> eigenbasis for A, the multiplicity for $\lambda = 1$ is two, but $\dim(E_1) = 1$ (there is only one eigenvector)

For E_{-2} we have situation similar to case I distinct real root.

$$C_3 e^{\lambda t} \Rightarrow \lambda = -2$$

$$C_3 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

In this we find our second eigenvector V_2 corresponds with $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

$$E_{-2} = \text{Span}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$A = 3\times3$ matrix but has only two eigenvectors

A does not have an eigenbasis.

we can also verify this with direct computation of matrix A.