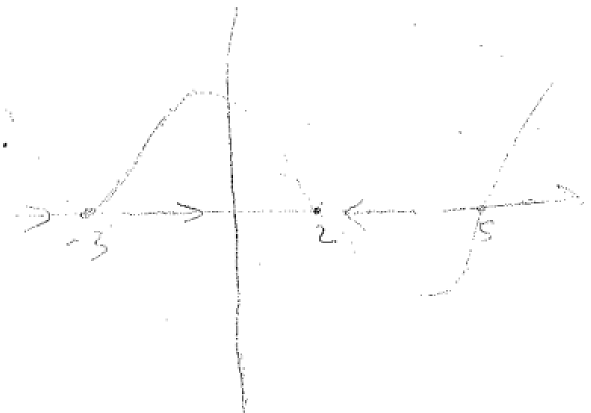
Exercise 1. (11pt)
Consider the autonomous first-order differential equation.

$$y' = (y+3)^2(y-2)(y-5)$$

(1) Draw the phase line. (3pt)

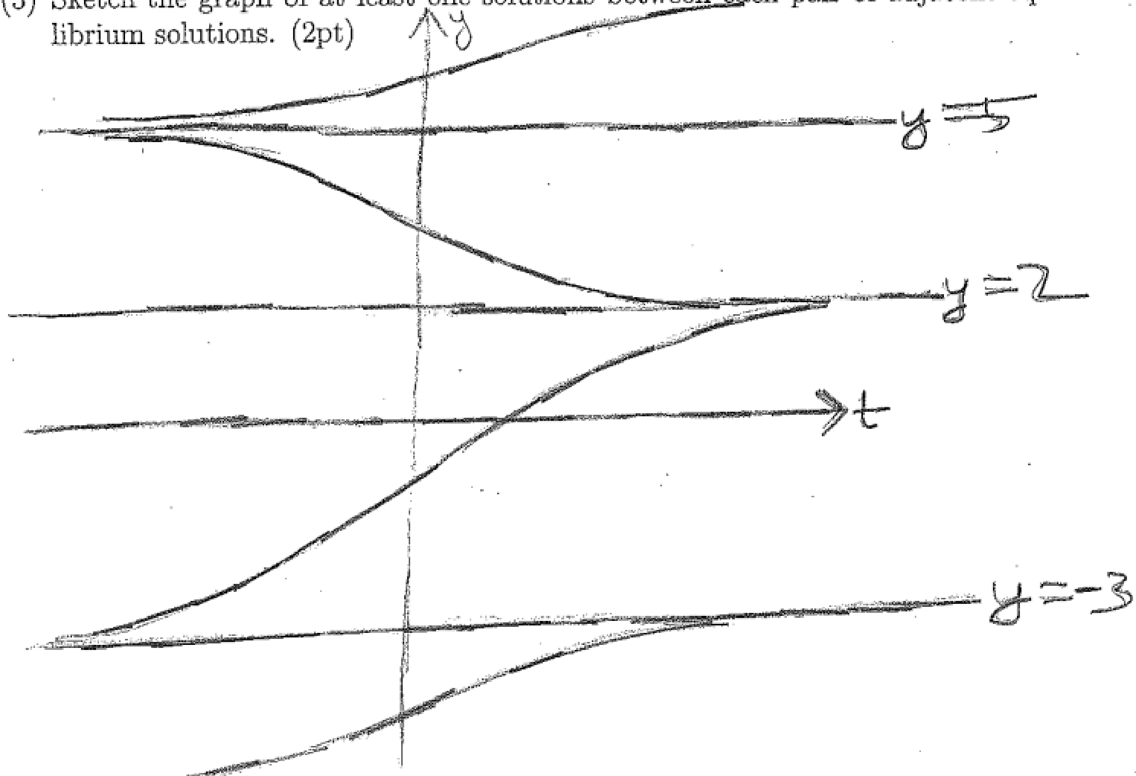


Phase Line

(2) What are the equilibrium solutions? Which are stable, and which are unstable? (3pt)

Equilibrium solutions	
$y = -3$	no conclusion
$y = 2$	stable
$y = 5$	unstable

(3) Sketch the graph of at least one solutions between each pair of adjacent equilibrium solutions. (2pt)



(4) Let $y_p(t)$ be a particular solution to the equation which satisfies $y_p(0) = 0$. Is it possible that $y_p(2) = 2$? Justify your answer. (3pt)

No, because $y_p(0)$ is in the interval $(-3, 2)$ and $y_p(2)$ does not exist on that interval.

Exercise 2. (8pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{\sqrt{x^2-4}}{t^2}.$$

(1) Can you apply the existence and uniqueness theorem to the initial value problem $x_0(1) = 6$? Justify your answer and give the biggest rectangle in which you can apply it containing the given point (if it exists). (5 pt)

$$\frac{dx}{dt} = \frac{\sqrt{x^2-4}}{t^2} \qquad t \neq 0$$

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{dt}{t^2}$$

$$\int \frac{dx}{\sqrt{x^2-4}} = -\frac{1}{t} + C$$

$(x^2-4)^{-1/2}$

Yes, because the function $x(t)$ and $\frac{dx}{dt}$ are both continuous at the point $(1,6)$.
The largest rectangle in which the theorem can be applied is $(0,\infty) \times (-16, 16)$.

(2) Can $x_0(2) = 5$ ($x_0(t)$ is the solution to the initial value problem in part 1))?(1pt) Justify your answer. (2pt)

No, because the slope at $x_0(1) = 6$ is positive, meaning that x gets larger as t does at this point.

Exercise 3. (6pt) Find a **particular** solution to the following differential equation

$$3y'' + 2y' - y = -4e^{3t}.$$

$y = ae^{3t}$

$y' = 3ae^{3t}$

$y'' = 9ae^{3t}$

$3y'' + 2y' - y = -4e^{3t}$

$27ae^{3t} + 6ae^{3t} - ae^{3t} = -4e^{3t}$

$32ae^{3t} = -4e^{3t}$

$a = -\frac{1}{8}$

$$\boxed{y_p = -\frac{1}{8}e^{3t}}$$

Exercise 4. (5pt) Consider the following problem:

$$y'' + y = 0 \qquad y(0) = 0 \;\; y'(\pi/2) = 0$$

(1) Show that $y(t) = C \cdot \sin(t)$ is a solution for any constant C. (3pt)

$y = C\sin t$

$y' = C\cos t$

$y'' = -C\sin t$

$\boxed{y'' + y = -C\sin t + C\sin t = 0}$ ✓

(2) Why does this not violate the 2. order existence and uniqueness theorem? (2pt)

It does not violate it because $y(t)$ and $y''(t)$ are continuous at all real numbers.

Exercise 5. (8pt) Consider the differential equation

$$y'' + \frac{1+x}{x}y' - \frac{1}{x}y = 0$$

(1) Check that $1+x$ and $\frac{2x^2+6x+4}{x+2}$ are solutions to the above equation. (4pt)

$1+x$:

$$y = 1+x$$
$$y' = 1$$
$$y'' = 0$$

$$y'' + \frac{1+x}{x}y' - \frac{1}{x}y = 0 + \frac{1+x}{x} - \frac{1+x}{x} = 0 \checkmark$$

$\frac{2x^2+6x+4}{x+2}$:

$$y = \frac{2x^2+6x+4}{x+2} = (2x^2+6x+4)(x+2)^{-1}$$

$$y' = (4x+6)(x+2)^{-1} - (2x^2+6x+4)(x+2)^{-2}$$

$$y'' = 4(x+2)^{-1} - (4x+6)(x+2)^{-2} - (4x+6)(x+2)^{-2} + 2(2x^2+6x+4)(x+2)^{-3}$$

$$y'' + \frac{1+x}{x}y' - \frac{1}{x}y = \frac{4}{x+2} - \frac{2(4x+6)}{(x+2)^2} + \frac{2(2x^2+6x+4)}{(x+2)^3} + \frac{1+x}{x}\left(\frac{4x+6}{(x+2)} - \frac{(2x^2+6x+4)}{(x+2)^2}\right) - \frac{1}{x}\frac{(2x^2+6x+4)}{x+2}$$

$$= \frac{4}{x+2} - \frac{8x+12}{(x+2)^2} + \frac{4x+4}{(x+2)^2} + \frac{(1+x)(4x+6)}{x(x+2)} - \frac{2x+2}{x+2} - \frac{(2x+2)}{x}$$

$$= \frac{4}{x+2} - \frac{4x+8}{(x+2)^2} + \frac{4x^2+10x+6}{x(x+2)} - \frac{2x+2}{x+2} - \frac{2x+2}{x}$$

$$= \frac{4x^2+8x-4x^2-8x+4x^3+16x^2+20x+12-(2x^2+2x)(x+2)-(2x+2)(x^2+4x+4)}{x(x+2)^2}$$

$$= \frac{\cancel{4x^3}+\cancel{18x^2}+20x+12-\cancel{2x^3}-\cancel{4x^2}-\cancel{2x^2}-4x-\cancel{2x^3}-\cancel{8x^2}-8x-\cancel{2x^2}-8x-8}{x(x+2)^2}$$

$$= 0 \checkmark$$

$$y' = 2$$
$$y'' = 0$$

$$0 + \frac{2(1+x)}{x} - \frac{2x^2+6x+4}{x(x+2)}$$
$$= 0 \checkmark$$

(2) Do they form a fundamental set of solutions?(1pt) Justify your answer. (2pt)

$y_1 = x+1 \qquad y_1' = 1$

$y_2 = \dfrac{2x^2+6x+4}{x+2} \qquad y_2' = \dfrac{(4x+6)}{x+2} - \dfrac{2x^2+6x+4}{(x+2)^2} = \dfrac{4x+6-2x-2}{x+2}$

$= 2$

$$W = \det\begin{vmatrix} x+1 & \dfrac{2x^2+6x+4}{x+2} \\ 1 & 2 \end{vmatrix} = 2(x+1) - \dfrac{2x^2+6x+4}{x+2}$$

$$= 2x+2 - (2x+2) = 0$$

$\boxed{\text{No, since } W = 0, \text{ and } y_2 \text{ is a constant multiple of } y_1}$

Exercise 6. (3pt)
Consider the second order equation

$$y'' - 2e^t y' - \tan(t)y = \sqrt{t^2+1}.$$

Write this equations as a planar system of first-order equations.

$$v = y'$$
$$v' = 2e^t v + \tan(t) y + \sqrt{t^2+1}$$