$V =$ ______

Exercise 1. (5pt + 3pt)
Consider the differential equations

$P(1,v)\quad Q(1,v)$

(1) 2 $$(y^2x + x)dx + (2yx^2 - 3y^2 + 2y)dy = 0$$

(2) 3 $$\frac{x^3+y^3}{x^2}dx + \frac{y^2+xy}{x}dy = 0$$

Find the integrating factor for the above equations.
(Hint for (1): it only depends on y)
(Hint for (2): Show first that it homogeneous for some degree n) $\frac{1+v^3}{v}$

1). find $u(y)$

$$\frac{\partial\left(P(x,y)\cdot u(y)\right)}{\partial y} = \frac{\partial\left(Q(x,y)\cdot u(y)\right)}{\partial x}$$

$$P'u + u'P = uQ'$$

$$u' = \frac{1}{P}u\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$u(y) = e^{-\int \frac{1}{P}\cancel{u}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4xy - 2yx = 2xy$$

$$\int 2xy\,dy = 2x\frac{y^2}{2} = 2xy^2$$

$$\boxed{u(y) = e^{\int \frac{2xy^2}{y^2x+x}} = e^{\int \frac{2y^2}{y^2+1}}}$$

2). $$P(tx,ty) = \frac{t^3x^3 + t^3y^3}{x^2t^2} = t\,P(x,y)$$

$$Q(tx,ty) = \frac{t^2y^2 + tx\,ty}{tx} = t\,Q(x,y)$$

hom... for...

$y = vx \qquad dy = v\,dx + x\,dv$

$$\frac{x^3+v^3x^3}{x^2}dx + \frac{v^2x^2+vx^2}{x}(v\,dx + x\,dv) = 0$$

$(x + v^3x \ldots$

Exercise 2. (12pt) Consider the differential equation

$$\frac{dy}{dx} = \frac{2(y+1)^2}{(x-1)^3}$$

(1) Find the explicit general solution. (4pt)

$$\int \frac{dx}{(x-1)^3} = \int \frac{dy}{2(y+1)^2}$$

-1

$$+\frac{1}{2}(x-1)^{-2} = \overset{1}{+2}(y+1)^{-1} + C$$

$$\frac{1}{(x-1)^2} = \frac{4}{y+1} + C$$

$$y = 4(x-1)^2 - 1 + C$$

-1

(2) Find the solution to this equation that satisfies the initial condition $y(0) = -2$. (3pt)

$$y(0) = 4 - 1 + C = -2$$

$$C = -5$$

$$y = 4(x-1)^2 - 6$$

(3) What is the interval of existence of the solution you found in (b). (3pt)

not constant 1

$x \neq 1$ ✓ $y \neq -1$

$4(x-1)^2 - 6 \neq -1 \Rightarrow x-1 \neq \pm \frac{\sqrt{5}}{2}$

$x \in \mathbb{R}, x \neq 1$,

y exists in $\cancel{(-6, \infty)}$

$x \neq 1$

1/3

(4) Find the solution to this equation that satisfies the initial condition $y(0) = -1$. (2pt)

$y(0) = 4(x-1)^2 - 1 + c$

$= -4 - 1 + c$

$= -5 + c = -1$

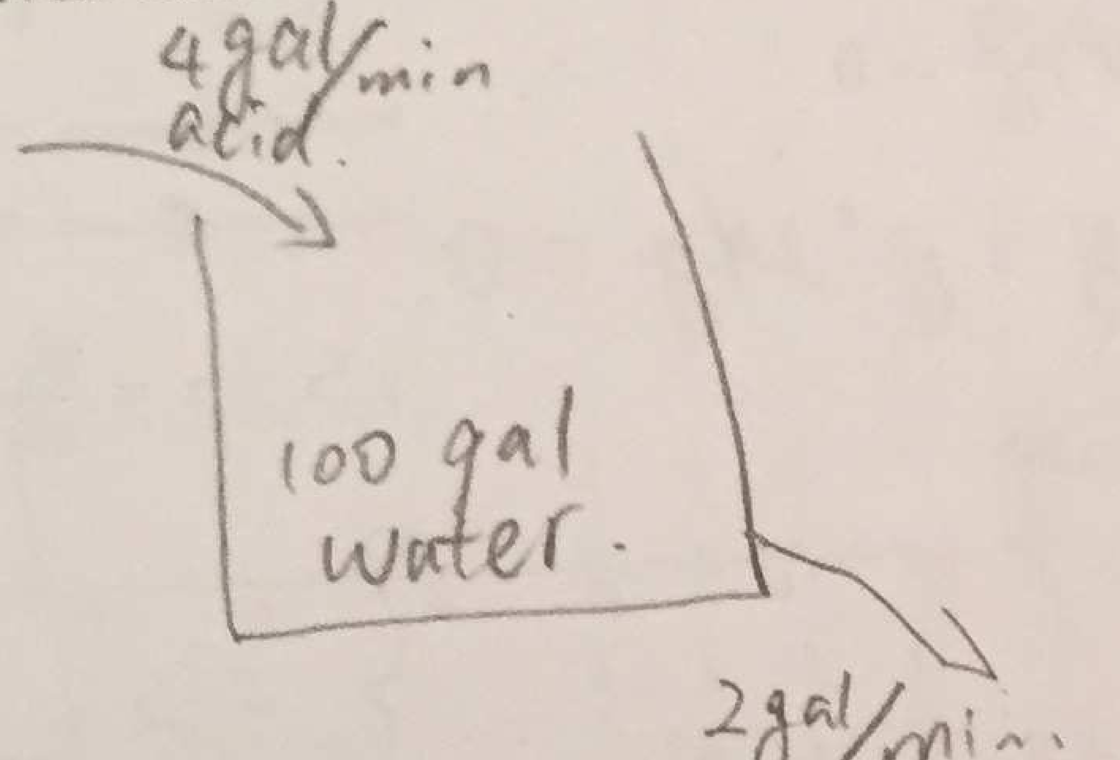
$c = 4$

$y = 4(x-1)^2 + 3$

-2

2

Exercise 3. (7pt) Suppose there is a tank filled with 100 gallons of water. Pure acid flows into the tank at a rate of 4 gal/min and the well mixed solution leaves the tank at the of 2 gal/min rate. Let $x(t)$ be the volume in gallons of acid in the tank at time t. Find $x(t)$ for any given time t.



$$x'(t) = \text{Inflow} - \text{outflow}$$

$$= 4 - \frac{2x(t)}{100+2t}$$

$$= 4 - \frac{x(t)}{100+t}$$

$$u(t) = e^{\int \frac{1}{100+t} dt} = 100+t$$

$$x(t) = \frac{1}{u(t)}\left(\int (100+t) \times 4 \, dt + C \right)$$

$$= \frac{1}{100+t}\left[4\left(100t + \frac{t^2}{2}\right) + C \right]$$

$$= \frac{400t + 2t^2}{\cancel{100}+t} + \frac{C}{\cancel{100}+t}$$

$$x(0) = \frac{C}{100} = 0 \qquad C = 0$$

$$x(t) = \frac{400t + 2t^2}{1\cancel{00}+t}$$

-1

Exercise 4. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$2(x + xy^2) + b(x^m y + y^2)\frac{dy}{dx} = 0$$

$2(x + xy^2)dx + b(x^m y + y^2)dy = 0.$

$$\left.\begin{array}{l} \frac{\partial P}{\partial y} = 4xy \\ \frac{\partial Q}{\partial x} = b(m x^{m-1} y) = bm\, x^{m-1} y \end{array}\right\} \Rightarrow \boxed{\begin{cases} b = 2 \\ m = 2 \end{cases}}$$

+3

(2) Solve the equation using the value of a and n you obtained in part (a). (6pt)

$F = 2(x + xy^2)dx + 2(x^2 y + y^2)dy = 0.$

$$F = 2\int x + xy^2 \, dx + \phi(y)$$

$$= 2\left(\frac{x^2}{2} + \frac{x^2}{2}y^2\right) + \phi(y) = x^2 + x^2y^2 + \phi(y).$$

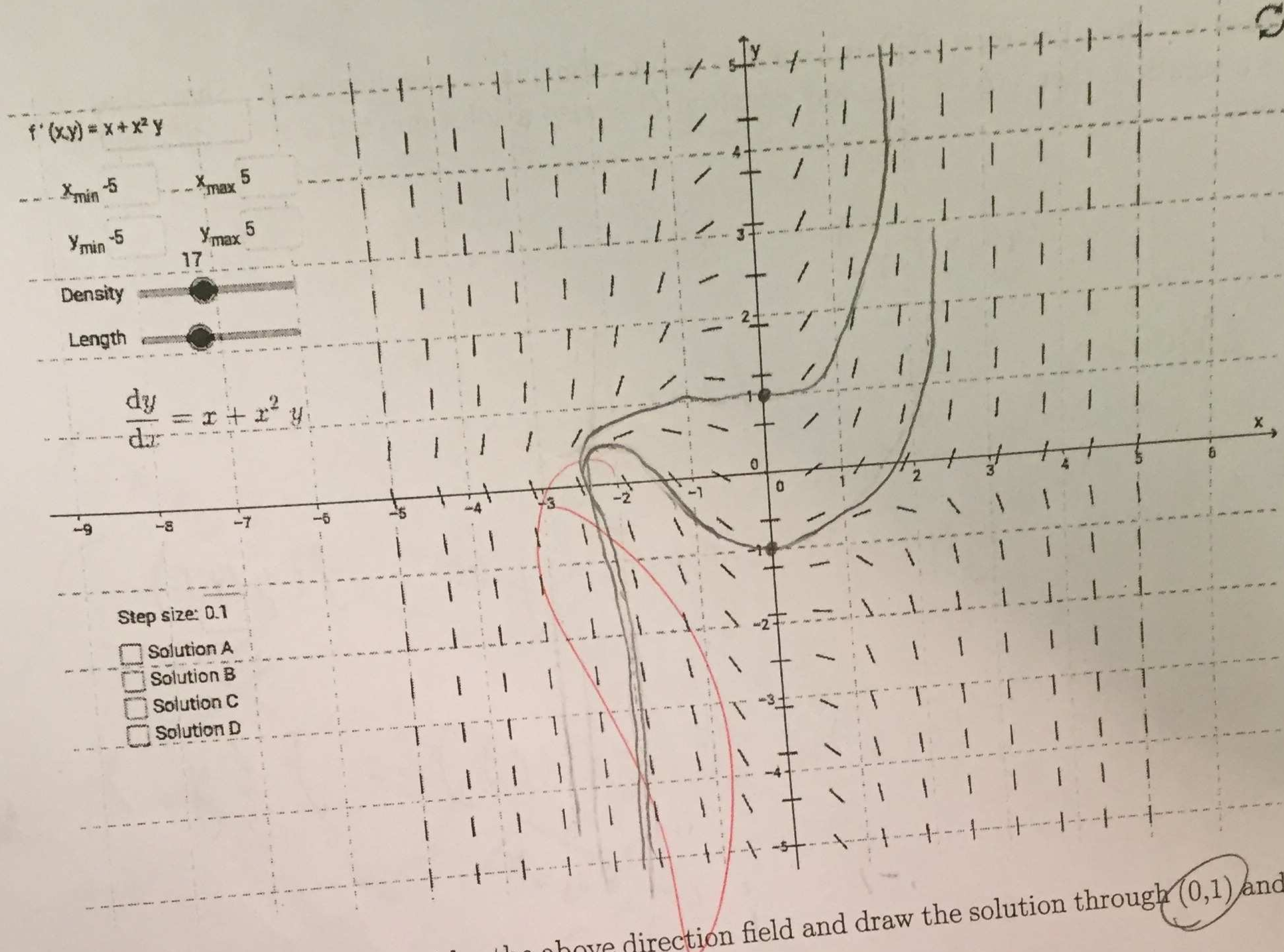
$$\frac{\partial F}{\partial y} = 2x^2 y + 2y^2 = 2yx^2 + \phi'(y)$$

$$\phi'(y) = 2y^2 \qquad \phi(y) = \frac{2}{3}y^3$$

$$\therefore F = x^2 + x^2y^2 + \frac{2}{3}y^3 = C$$

+6

9



Exercise 5. (4pt) Consider the above direction field and draw the solution through (0,1) and (0,-1).

2

Exercise 6. (3pt) Let $y' = f(y)$, where f is a function just depending on y. Show that if $y(t)$ is a solution, then $y(t+C)$ for any constant C is also a solution to the given differential equation.

let $y_2 = y(t+C)$.

Show : $y_2' = f(y_2)$.

Since $y(t)$ is solution, $y'(t) = f(y(t))$

$y_2' = y'(t+c) = f(y(t+c)) = f(y_2)$

2-