MIDTERM 1

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Math33B
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Problem	Points	Score
1	8	
2	12	
3	7	
4	7	
SA	6	
Total	40	

Exercise 1. (8pt)
Consider the differential equations

$$2y^2 + 4x^2 + 2xy\frac{dy}{dx} = 0$$

$2y^2x + 4x^3 + 2x^2y$

$4yx \quad 4xy$

(1) Find the integrating factor for the above equations. (4pt)
(Hint: it only depends on x)

$$(2y^2+4x^2)\,dx + 2xy\,dy = 0$$

$$h = \frac{1}{2xy}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{2xy}(4y - 2y)$$

$$= \frac{2}{x} - \frac{1}{x} = \frac{1}{x}$$

$$M = e^{\int \frac{1}{x}} = e^{\ln x} = x$$

$$\boxed{M = x}$$

(2) Solve the equation. (4pt)

$$(2y^2x + 4x^3)\,dx + 2x^2y\,dy = 0$$

$$F = \int 2y^2x + 4x^3\,dx + \phi(y)$$

$$= y^2x^2 + x^4 + \phi(y)$$

$$\frac{dF}{dy} = 2yx^2 + \phi'(y) = 2x^2y$$

$$\phi'(y) = 0$$

$$\phi'(y) = 0 \longrightarrow F(x,y) = y^2x^2 + x^4 = C$$

$$\boxed{y^2x^2 + x^4 = C}$$

$$\frac{1}{y} + \frac{1}{y-1} = \frac{\quad}{y(y-1)}$$

$$\frac{1}{y(y-1)} = \frac{-1}{y} + \frac{+1}{y-1}$$

Exercise 2. (12pt) Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2 - y}{x}$$

(1) Find the explicit general solution. (5pt)

$$\frac{dy}{y^2-y} = \frac{dx}{x}$$

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{y-1} - \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y-1| - \ln|y| = \ln|x| + C$$

$$\ln\left|\frac{y-1}{y}\right| = \ln|x| + C$$

$$1 - \frac{1}{y} = Ce^{\ln x} = Cx$$

$$\frac{1}{y} = 1 - Cx \qquad \boxed{y = \frac{1}{1-Cx}}$$

(2) Find the solution to this equation that satisfies the initial condition $y(1) = 2$. (2pt)

$$y(1) = 2$$

$$2 = \frac{1}{1-C(1)} \rightarrow C = \frac{1}{2}$$

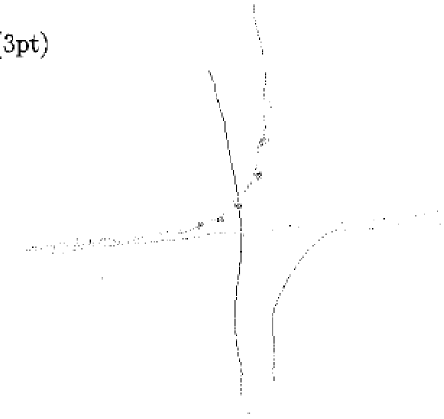
$$y = \frac{1}{1-\frac{1}{2}x}$$

$$\boxed{y = \frac{2}{2-x}}$$

(3) What is the interval of existence of the solution you found in (b). (3pt)

$y = \frac{2}{2-x}$

$x \neq 2$ $\quad$ $y(1) = 2$



$\boxed{\text{IoE: } (-\infty, 2)}$

(4) Find the solution to this equation that satisfies the initial condition $y(1) = 0$. (2pt)

$y(1) = 0$

$y = \frac{1}{1-Cx} \neq 0$

$\boxed{y(x) = 0}$

Exercise 3. (7pt) Suppose there is a tank filled with 100 gallons of water. Pure acid flows into the tank at a rate of 4 gal/min and the well mixed solution leaves the tank at the of 2 gal/min rate. Let $x(t)$ be the volume in gallons of acid in the tank at time t. Find $x(t)$ for any given time t.

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$= 4 - \frac{2x(t)}{100+2t} = 4 - \frac{x(t)}{50+t}$$

$$\frac{dx}{dt} + \frac{x}{50+t} = 4$$

$$u = e^{\int \frac{1}{50+t}dt} = e^{\ln|t+50|} = t+50$$

$$(t+50)x = \int 4t+200 = 2t^2+200t+C$$

$$x = \frac{2t^2+200t+C}{t+50}$$

$$x(0) = 0$$

$$x = \frac{2t^2+200t}{t+50}$$

$$\boxed{x(t) = \frac{2t^2+200t}{t+50}}$$

Exercise 4. (7pt) Consider

$$4yxdx + 5x^2dy$$

(1) Show that the above equation is not exact. (3pt)

$$\frac{\partial P}{\partial y} = 4x$$

$$\frac{\partial Q}{\partial x} = 10x$$

$\boxed{4x \neq 10x \quad \text{therefore not exact}}$

(2) Find a and b such that $x^a y^b$ is an integration factor of the above equation. (4pt)

$$4xy\,dx + 5x^2dy$$

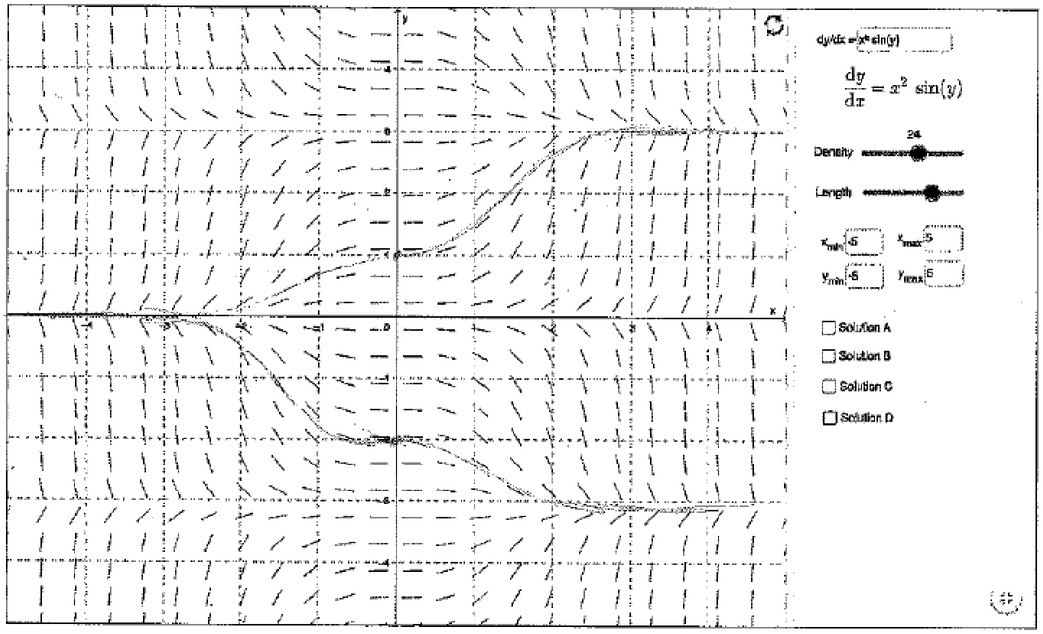
$$\div y \qquad \div x^2$$

$$\mu = \frac{1}{x^2 y} = x^{-2}y^{-1}$$

$$\frac{4}{x}dx + \frac{5}{y}dy \rightarrow \frac{\partial P}{\partial y} = 0 \quad \frac{\partial Q}{\partial x} = 0 \quad \checkmark$$

$$\boxed{\begin{array}{l} a = -2 \\ b = -1 \end{array}}$$

Field M1 F18.png



1. Short answer problems

(no explanation needed)

(1) (4pt) Consider the above direction field and draw the solution through (0,1) and the solution through (0,-2).

(2) (2pt) Which of the following are homogeneous differential equations?

Y / N $\sin(\frac{x}{y})dy + 2dx = 0$ Q: 0 P: 0

Y / N $(xy + x^2)dy + (y^2x - x^2y)dx$ Q: 2 P: 2

Y / N $\sin(xy)dy - \cos(xy)dx$ Q: 2 P: 2

Y / N $\sqrt{x^2y^2 - 4xy^3}dy + x^2dx$ Q: 2 P: 2