FINAL

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Math33B
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section: 2C

Problem	Points	Score
1	7	
2	5	
3	10	
4	10	
5	8	
6	9	
7	9	
8	8	
9	9	
10	9	
11	11	
12	5	
Total	100	

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Solve the following equation. (*Hint: Find the integrating factor*)

$$(x^2 + y^2)\,dx - 2xy\,dy = 0$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = -2y$$

$2y \neq -2y$ ∴ not exact

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -4y$$

$$h = \frac{1}{Q}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{-2xy} \cdot -4y = \frac{2}{x}$$

$$u(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$(x^4 + x^2y^2)\,dx + (-2x^3y)\,dy = 0$$

$$F(x,y) = \int (x^4 + x^2y^2)\,dx + \Phi(y)$$

$$F(x,y) = \frac{x^5}{5} + \frac{x^3y^2}{3} + \Phi(y)$$

$$\frac{\partial F}{\partial y} = \frac{2x^3y}{3} + \Phi'(y) = -2x^3y$$

$$\Phi'(y) = -\frac{6x^3y}{3} - \frac{2x^3y}{3}$$

$$\Phi'(y) = -\frac{8x^3y}{3}$$

$$\Phi(y) = -\frac{4x^3y^2}{3}$$

$$F(x,y) = \frac{x^5}{5} + \frac{x^3y^2}{3} - \frac{4x^3y^2}{3} = 0$$

$$F(x,y) = \frac{x^5}{5} - x^3y^2 = 0$$

Exercise 2. (5pt) Solve $y' = y(y+1)(x+2)(x+3)$

$$\frac{dy}{y(y+1)} = \frac{dx}{(x+2)(x+3)}$$

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \qquad 1 = Ay + A + By \qquad A = 1,\ B = -1$$

$$\frac{1}{(x+2)(x+3)} = \frac{C}{x+2} + \frac{D}{x+3} \qquad 1 = Cx + 3C + Dx + 2D \qquad C = -D$$

$$C + D = 0$$

$$3C + 2D = 1$$

$$3C + 2(-C) = 1$$

$$C = 1$$

$$D = -1$$

$y \neq 0$, $y \neq -1$, $x \neq -2$, $x \neq -3$

$$\frac{dy}{y} - \frac{dy}{y+1} = \frac{dx}{x+2} - \frac{dx}{x+3}$$

$$\int \frac{dy}{y} - \int \frac{dy}{y+1} = \int \frac{dx}{x+2} - \int \frac{dx}{x+3}$$

$$\ln|y| - \ln|y+1| = \ln|x+2| - \ln|x+3| + c$$

$$\ln\left|\frac{y}{y+1}\right| = \ln\left|\frac{x+2}{x+3}\right| + C$$

$$e^{\ln\left|\frac{y}{y+1}\right|} = e^{\ln\left|\frac{x+2}{x+3}\right| + C} \qquad A = \pm C$$

$$\left|\frac{y}{y+1}\right| = C\left|\frac{x+2}{x+3}\right| \qquad 1 + \frac{1}{y} = A\left(\frac{x+3}{x+2}\right)$$

flip both sides

$$\frac{y}{y+1} = A\left(\frac{x+2}{x+3}\right) \qquad \frac{1}{y} = A\left(\frac{x+3}{x+2}\right) - 1$$

$$\frac{y+1}{y} = A\left(\frac{x+3}{x+2}\right) \qquad \boxed{y(x) = \frac{1}{A\left(\frac{x+3}{x+2}\right) - 1}}$$

Exercise 3. (10pt) Find a **particular** solution to the following two differential equations

(1) $y'' + 4y = 8t^2 - 4t$ (2pt)

Guess: $y = At^2 + Bt + C$

$y' = 2At + B$

$y'' = 2A$

$\boxed{y_p(t) = 2t^2 - t - 1}$

$2A + 4(At^2 + Bt + C) = 8t^2 - 4t$

$4At^2 + 4Bt + 4C + 2A = 8t^2 - 4t$

$A = 2 \quad B = -1 \quad C = -1$

$4C + 2(2) = 0$
$4C = -4$
$C = -1$

(2) $y'' + 4y = 4\sin(2t)$ (4pt)

Guess: $y = A\cos(2t) + B\sin(2t)$

$y' = -2A\sin(2t) + 2B\cos(2t)$

$y'' = -4A\cos(2t) - 4B\sin(2t)$

$-4A\cos(2t) - 4B\sin(2t) + 4A\cos(2t) + 4B\sin(2t) = 4\sin(2t)$

$0 = 4\sin(2t)$

Guess 2: $y = At\cos(2t) + Bt\sin(2t)$

$y' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$

$y'' = -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t)$
$+ 2B\cos(2t) - 4Bt\sin(2t)$

$y'' = (4B - 4At)\cos(2t) + (-4Bt - 4A)\sin(2t)$

$(4B - 4At)\cos(2t) + (-4Bt - 4A)\sin(2t) + 4At\cos(2t) + 4Bt\sin(2t) = 4\sin(2t)$

$4B\cos(2t) - 4A\sin(2t) = 4\sin(2t)$

$B = 0 \quad A = -1$

$y_p(t) = -t\cos(2t)$

(3) Give the **general** solution to the following differential equation

$$y'' + 4y = 8\sin(2t) - 8t^2 + 4t. \text{ (4pt)}$$

$\lambda^2 + 4 = 0$ – solution to homogenous

$\lambda = \pm 2i \rightarrow$ $y_h(t) = C_1\cos(2t) + C_2\sin(2t)$

for $y'' + 4y = 8\sin(2t)$

$y_p(t) = -2t\cos(t)$

for $y'' + 4y = -8t^2 + 4t$

$y_p(t) = -2t^2 + t + 1$

therefore the general solution is

$y(t) = -2t\cos(t) - 2t^2 + t + 1 + C_1\cos(2t) + C_2\sin(2t)$

Exercise 4. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 1 \\ -6 \\ -2 \end{pmatrix}$$

Since upper triangular

$\lambda_1 = 3$ almult 1 $\quad \lambda_2 = -1$ almult 3

$$A - 3I = \begin{pmatrix} -4 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 8 \\ 0 \end{pmatrix} \qquad \vec{y}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \\ 8 \\ 0 \end{pmatrix}$$

$$A + I = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

one free variable → geomult = 1

$$(A + I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 8 & -6 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 64 & -48 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\ker\left((A+I)^3\right) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix} \right)$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix}$$

this term will always truncate since we use vector in its kernel

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$$e^{At} = e^{\lambda t} e^{(A-\lambda I)t}$$

$$y_2(t) = e^{At} v_2 = e^{-t} e^{(A+1)t} v_2 = e^{-t}\left(v_2 + t(A+1)v_2 + \frac{t^2}{2}(A+1)^2 v_2 + \frac{t^3}{6}(A+1)^3 v_2\right)$$

When $v_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $(A+1)v_2 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

so $$y_2(t) = e^{-t}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right)$$

$$y_2(t) = e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$y_3(t) = e^{At} v_3 = e^{-t} e^{(A+1)t} v_3 = e^{-t}\left(v_3 + t(A+1)v_3 + \frac{t^2}{2}(A+1)^2 v_3\right)$$

$$= e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}0&2&0&0\\0&0&1&0\\0&0&4&-3\\0&0&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0&0&2&0\\0&0&4&-3\\0&0&16&-12\\0&0&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)$$

$$= e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right)$$

$$y_3(t) = e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\0\\0\end{pmatrix}\right)$$

$$y_4(t) = e^{-t}\left(v_4 + t(A+1)v_4 + \frac{t^2}{2}(A+1)^2 v_4\right)$$

$$= e^{-t}\left(\begin{pmatrix}0\\0\\6\\8\end{pmatrix} + t\begin{pmatrix}0&2&0&0\\0&0&1&0\\0&0&4&-3\\0&0&0&0\end{pmatrix}\begin{pmatrix}0\\0\\6\\8\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0&0&2&0\\0&0&4&-3\\0&0&16&-12\\0&0&0&0\end{pmatrix}\begin{pmatrix}0\\0\\6\\8\end{pmatrix}\right)$$

$$= e^{-t}\left(\begin{pmatrix}0\\0\\6\\8\end{pmatrix} + t\begin{pmatrix}0\\6\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}12\\0\\0\\0\end{pmatrix}\right)$$

$$y(t) = c_1 e^{3t}\begin{pmatrix}1\\2\\8\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}2t\\1\\0\\0\end{pmatrix} + c_4 e^{-t}\begin{pmatrix}6t^2\\6t\\6\\8\end{pmatrix}$$

$$\begin{pmatrix}3\\1\\-6\\-2\end{pmatrix} = c_1\begin{pmatrix}1\\2\\8\\0\end{pmatrix} + c_2\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\6\\8\end{pmatrix}$$

$c_1 + c_2 = 3$

$2c_1 + c_3 = 1$

$8c_1 + 6c_4 = -6$

$8c_4 = -2$

$c_4 = -\frac{1}{4}$

$8c_1 - \frac{3}{2} = \frac{-12}{2}$

$8c_1 = \frac{-9}{2}$; $c_1 = \frac{-9}{16}$

$c_2 - \frac{9}{16} = \frac{48}{16}$

$c_2 = \frac{57}{16}$

$\frac{-18}{16} + c_3 = \frac{16}{16}$

$c_3 = \frac{34}{16}$

Solution to initial value problem:

$$\vec{y}(t) = \frac{-9}{16}e^{3t}\begin{pmatrix}1\\2\\8\\0\end{pmatrix} + \frac{57}{16}e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{34}{16}e^{-t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \frac{1}{4}e^{-t}\begin{pmatrix}0\\0\\6\\8\end{pmatrix}$$

$2t^3\ 3t^2+6t+4t^2$

Exercise 5. (8pt) Consider the differential equation

$$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t-1) - t(e^t+1), \qquad (t>0)$$

(1) Show that $y_1 = e^t(2t+1) - (t+1)$ is solutions to the above equation. (4pt)
(Show ALL your calculations in detail for full credit)

$$y_1 = 2te^t + e^t - t - 1 = (2t+1)e^t - (t+1)$$

$$y_1' = 2e^t + 2te^t + e^t - 1 = (3+2t)e^t - 1$$

$$y_1'' = 2e^t + 2e^t + 2te^t + e^t = (5+2t)e^t$$

$$t^2(5+2t)e^t - (t^2+2t)(3+2t)e^t + (t^2+2t) + (t+2)y =$$

$$(5t^2+2t^3)e^t - (2t^3+7t^2+6t)e^t + (t^2+2t)$$
$$+ (t+2)\left((2t+1)e^t - (t+1)\right) =$$

$$(-2t^2-6t)e^t + (t^2+2t) + (2t^2+5t+2)e^t - (t^2+3t+2) =$$

$$(-t+2)e^t - t - 2 = 2(e^t-1) - t(e^t+1)$$

$$-te^t + 2e^t - t - 2 = 2e^t - 2 - te^t - t$$

$$0 = 0$$

$e^t - te^t + 2t - 1$

$e^t t + e^t + t - 1$

$-te^t \quad e^t \quad + 2t - 1$

(2) Given that $y_2 = e^t(t+1)+(t-1)$, and $y_3 = e^t(1-t)+(2t-1)$ are also solutions to the above equation, find the general solution to the equation. Justify your answer. (4pt)

$$Y(t) = C_1\left(e^t(2t+1) - (t+1)\right) + C_2\left(e^t(t+1) + (t-1)\right)$$

- to make sure this is a proper general solution, we must see if ~~y_1, y_2, y_3~~ two of the solutions form a fundamental set by testing their linear independency

- By looking at the 3 functions, the Wronskian's can be estimated to not be 0, meaning y_1 and y_2 are linearly independent and the above equation is the general solution

$y_2' = e^t + te^t + e^t + 1$

$(2+t)e^t + 1$

$\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$

$\det\begin{pmatrix} y_1 & y_3 \\ y_1' & y_3' \end{pmatrix} \neq 0$ $\quad \det\begin{pmatrix} (2t+1)e^t - (t+1) & te^t + e^t + t - 1 \\ (3+2t)e^t - 1 & (2+t)e^t + 1 \end{pmatrix} \neq 0$

$\det\begin{pmatrix} y_2 & y_3 \\ y_2' & y_3' \end{pmatrix}$ $\quad \det\begin{pmatrix} (t+1)e^t + (t-1) & y_3 \\ (t+1)e^t + e^t & y_3' \end{pmatrix} = 0$

Exercise 6. (9pt) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (5pt)

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) + 2 = 3 - 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\frac{4 \pm \sqrt{16-20}}{2} \qquad \lambda = 2 \pm i$$

$$A - (2+i) = \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}$$

$$-x(i+1) - 2(\quad) = 0 \qquad \begin{pmatrix} -2 \\ i+1 \end{pmatrix}$$

$$-2 + (1-i)(1+i) = 0$$

We find $\begin{pmatrix} -2 \\ i+1 \end{pmatrix}$ is an eigenvector for $\lambda = 2+i$

$$e^{\lambda t} = e^{(2+i)t} = e^{2t} e^{it} = e^{2t}(\cos(t) + i\sin(t))$$

$$e^{\lambda t} v = e^{2t}(\cos(t) + i\sin(t))\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= e^{2t}\left(\begin{pmatrix} -2\cos(t) \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} -2\sin(t) \\ \sin(t) \end{pmatrix} + i\begin{pmatrix} 0 \\ \cos t \end{pmatrix} - \begin{pmatrix} 0 \\ \sin t \end{pmatrix}\right)$$

$$= e^{2t}\begin{pmatrix} -2\cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} + i\begin{pmatrix} -2\sin(t) \\ \sin(t) + \cos(t) \end{pmatrix}$$

$$Y(t) = C_1 e^{2t}\begin{pmatrix} -2\cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -2\sin(t) \\ \sin(t) + \cos(t) \end{pmatrix}$$

(2) Conclude that the equilibrium point is a spiral. (1pt)

– Since the λ's are in the form $a \pm bi$, and $a > 0$, the equilibrium point is a spiral

(3) Is it a sink or a source? (1pt)

Since the real part of λ is positive, it is a spiral <u>source</u>

(4) Does the spiral rotate clockwise or counterclockwise? (2pt)

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

counterclockwise

Prob diag 2018.pdf

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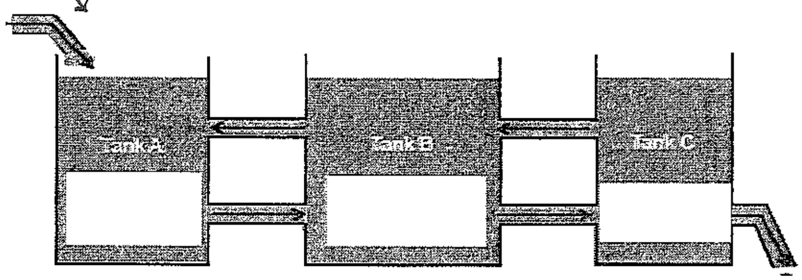


Exercise 7. (9pt)
Consider the above mixing problem with the following data.

- at time $t = 0$ there is 0 lb of salt in tank A, 10 lb of salt in tank B, and 20 lb of salt in tank C.
- Tank A contains initially 60 gallons of solution, Tank B contains initially 120 gallons of solution and Tank C contains initially 30 gallons of solution.
- 6 gal/min of water enters tank A through the top far left pipe.
- The solutions flows at
  - at 6 gal/min through the upper left pipe
  - at 12 gal/min through the lower left pipe
  - at 3 gal/min through the upper right pipe
  - at 9 gal/min through the lower right pipe
- 6 gal/min of solutions leaves tank C through the bottom far right pipe.

Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ in tank A, B, and C at time t (**you do NOT have to solve it!**).

$$\vec{X}(t) = \begin{pmatrix} X_a(t) \\ X_b(t) \\ X_c(t) \end{pmatrix}$$

A = amount of salt in Tank A
B = amount of salt in Tank B
C = amount of salt in Tank C

TANK A

$V_A(t) = 6 + 6 - 12 = 0$ — constant volume

TANK B

$V_b(t) = 3 + 12 - 6 - 9 = 0$ — constant volume

TANK C

$V_c(t) = 9 - 6 - 3 = 0$ — constant volume

$$X_a'(t) = \text{Rate IN} - \text{Rate OUT}$$
$$= 6\left(\frac{B}{120}\right) - 12\left(\frac{A}{60}\right) = \frac{1}{20}B - \frac{1}{5}A$$

$$X_b'(t) = \text{Rate IN} - \text{Rate OUT}$$
$$= 3\left(\frac{C}{30}\right) + 12\left(\frac{A}{60}\right) - 6\left(\frac{B}{120}\right) - 9\left(\frac{B}{120}\right) = \frac{1}{10}C + \frac{1}{5}A - \frac{1}{8}B$$

$$X_c'(t) = \text{Rate IN} - \text{Rate OUT}$$
$$= 9\left(\frac{B}{120}\right) - 9\left(\frac{C}{30}\right) = \frac{3B}{40} - \frac{3C}{10}$$

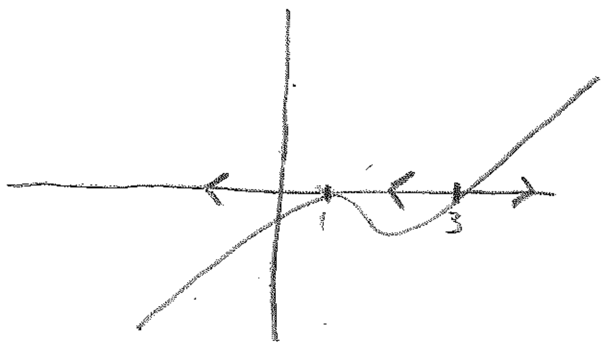
$$\vec{x}'(t) = \begin{pmatrix} -\frac{1}{5} & \frac{1}{20} & 0 \\ \frac{1}{5} & -\frac{1}{8} & \frac{1}{10} \\ 0 & \frac{3}{40} & -\frac{3}{10} \end{pmatrix} \vec{x}(t) \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}$$

$(x-1)\sqrt{x^2 -}$

Exercise 8. (8pt)
Consider the differential equation

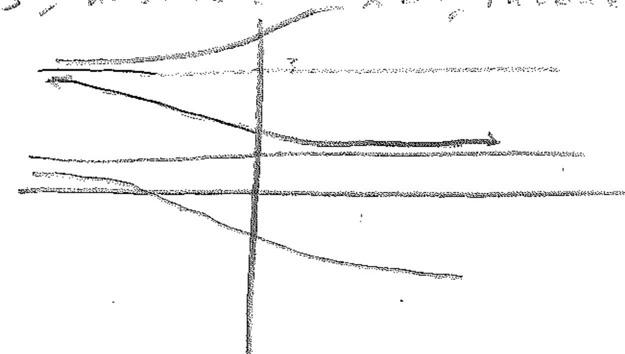
$$\frac{dx}{dt} = e^x(x^3 - 5x^2 + 7x - 3) = e^x(x-1)^2(x-3)$$

(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt) Eq points: $x = 1$, $x = 3$



(2) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)

$x = 3$, unstable $x = 1$, inconclusive



(3) Does there exist a solution of the equation, $x(t)$, satisfying $x(0) = -1$ and $x(2) = 0$? Justify your answer. (2pt)

if $x < 3$, then $e^x(x-1)^2(x-3) < 0$,
therefore $\frac{dx}{dt}$ is negative for all values of $x < 3$.

Since $x(0) = -1$ and $x(2) = 0$, $\frac{dx}{dt}$ would need to be positive at some point in the interval of x $[0, 2]$. Since $\frac{dx}{dt}$ is negative for all values of x in between 0 and 2, it is impossible for a solution of the equation $x(t)$ to satisfy $x(0) = -1$ and $x(2) = 0$

$\frac{1}{t}x - \frac{3}{t} + 2t^{-1}x^{-1}$

$t \neq 0$
$x \neq 0$

Exercise 9. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx} \qquad \frac{(x-2)(x-1)}{tx}$$

(1) Find all constant solutions of the above equation. (4pt)

$x(t) = 2$

$x(t) = 1$

(2) Argue that the range of the solution to the initial value problem $x(1) = 1.2$ is contained in $(1, 2)$. (3pt)

$$\frac{\partial F}{\partial x} = \frac{1}{t} - 2t^{-1}x^{-2} = \frac{1}{t} - \frac{2}{tx^2}$$ continuous as long as $t \neq 0$ and $x \neq 0$

- For this differential equation, existence and uniqueness can be applied on the rectangle $t: (0, \infty)$, $x: (0, \infty)$
- This means every point in the rectangle has a unique solution. Since $x = 2$ and $x = 1$ are both constant solutions, the solution to $x(1) = 1.2$ may not intersect either $x = 1$ or $x = 2$, thus its range is contained in $(1, 2)$

(3) Can you apply the existence theorem to the initial value problem $z(0) = 5$? (1pt) Justify your answer. (1pt)

X

- You may not apply the existence theorem because $\frac{dx}{dt}$ is not continuous for $x(0) = 5$ because $t \neq 0$

Exercise 10. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$2(x + xy^2) + b(x^m y + y^2)\frac{dy}{dx} = 0$$

$\frac{\partial P}{\partial y} = 4xy$

$\frac{\partial Q}{\partial x} = mbx^{m-1}y$

$4xy = mbx^{m-1}y$

$m = 2$

$b = 2$

(2) Solve the equation using the value of b and m you obtained in part (a). (6pt)

$(2x + 2xy^2)dx + (2x^2y + 2y^2)dy = 0$

$F(x, y) = \int (2x + 2xy^2)\,dx + \Phi(y)$

$= x^2 + x^2y^2 + \Phi(y)$

$\frac{\partial F}{\partial y} = 2x^2y + \Phi'(y) = 2x^2y + 2y^2$

$\Phi'(y) = 2y^2$

$\Phi(y) = \frac{2}{3}y^3$

$F(x, y) = x^2 + x^2y^2 + \frac{2}{3}y^3 = 0$

$T = \lambda_1 + \lambda_2$
$D = \lambda_1 \lambda_2$

Exercise 11. (11pt) Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$
$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(1) Determine where in the trace-determinante plane the system $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$ fit. (3pt)

$\begin{pmatrix} 3-\lambda & -2 \\ 0 & 3-\lambda \end{pmatrix} \to (3-\lambda)^2 = 0$ $\quad T = 6 \quad D = 9$

$\lambda = 3$ double 2 $\quad T^2 - 4D$

$36 - 36 = 0$

$\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \to (2-\lambda)^2 = 0$ $\quad T = 4 \quad D = 4$

$\lambda = 2$ double 2 $\quad T^2 - 4D$

$16 - 16 = 0$

Since $T^2 - 4D = 0$ for both $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$, they both lie directly on the $T^2 - 4D$ curve in Quadrant I since $T > 0$ and $D > 0$ for both cases

(2) Find all of the half line solutions for the system $\vec{y}' = A\vec{y}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

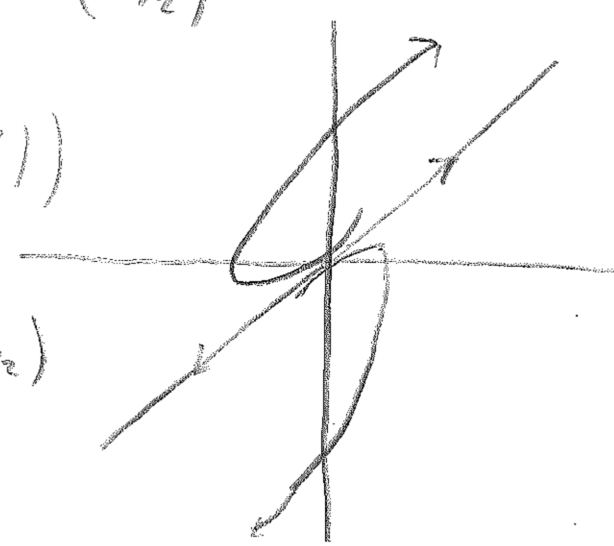
$A - 3I = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$

degenerate node



$y_1(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y_2(t) = e^{3t}\left(\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

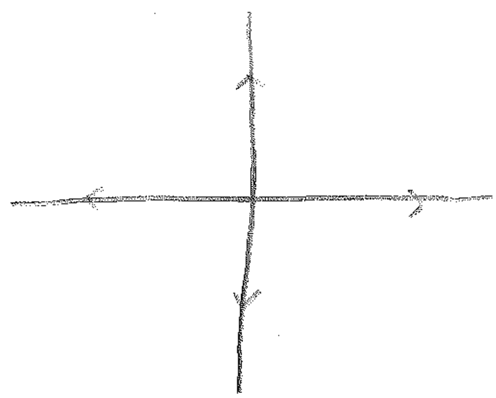
$y(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} t \\ -1/2 \end{pmatrix}$

$y(t) = e^{3t}(c_1 + c_2 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t} c_2 \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$

(3) Find all of the half line solutions for the system $\vec{x}' = B\vec{x}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$v' = y''$

Exercise 12. (5pt)

(1) Consider the second order equation $y'' + 3t^2y' - \cos(t)y = -3e^t$. Write this equations as a planar system of first-order equations. (2pt)

$v = y'$ $\quad v' + 3t^2 v - \cos(t)y = -3e^t$

$v' = \cos(t)y - 3e^t - 3t^2 v$

$F(t, y, y')$

(2) Consider more generally an n-order equation $y^{(n)} = F(t, y, \dots, y^{(n-1)})$. How can you write this as a system of first-order equations? (3pt)

$y^{4} = F(t, y, y^2, y^3)$

$v = y'''$

$v' = F(t, y, y^2, v)$

$x = y''$

$x' = y'''$

$v' = F(t, y, x, v)$

when $n = 4$ ∴ v and x are needed

if $n = 3$ $\quad y''' = F(t, y, y'')$

when $n = 3$ $\quad v = y''$

$v' = F(t, y, v)$

For $n > 2$, By induction, $y^{(n)} = F(t, y, \dots, y^{(n-1)})$ can be written as a system of first-order equations with the introduction of at least $(n-2)$ new variables. Part 1 shows one variable is needed when $n = 2$ and $n = 1$ is by definition first order

$$(x-1)(x-1)(x-3)$$

Extra page

$$\begin{array}{r} x^2 - 4x + 3 \\ x - 1 \enclose{longdiv}{x^3 - 5x^2 + 7x - 3} \\ \underline{-x^3 + x^2} \\ -4x^2 + 7x \\ \underline{+4x^2 - 4x} \\ 3x - 3 \end{array}$$

$$(A+1)^3 \begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix} = \vec{0}$$

$$(A+1)^2 \begin{pmatrix} 0 \\ 0 \\ 6 \\ 6 \end{pmatrix} \neq \vec{0}$$

thus fundamental set

$$\begin{pmatrix} 1 \\ 2 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix}, (A+1)\begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix}, (A+1)^2\begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$