Haofei Fan | Math 33B | Wed, Jan 29, 2020

Midterm 1

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Section:

	Tuesday:	Thursday:	
	1A	1B	TA: YIH, SAMUEL
	1C	1D	TA: KIM, BOHYUN
	1E	1F	TA: BOSCHERT, NICHOLAS

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you.. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please **circle or box your final answers.**

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
Total:	50	

1. (10 points) Solve the initial value problem:

$$x^2y' + 2xy + 1 = 0, y(1) = 0$$

$y' = \frac{-1-2xy}{x^2} = \frac{-1}{x^2} - \frac{2}{x}y$

$y' + \frac{2}{x}y = \frac{-1}{x^2}$

IF: $e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln|x^2|} = x^2$

$x^2\left(y' + \frac{2}{x}y\right) = \frac{-1}{x^2} \cdot x^2$

$\Downarrow$

$\int (x^2y)' = \int 1 \, dx \Rightarrow \frac{x^2y}{x^2} = \frac{-x+C}{x^2}$

$y = \frac{-1}{x} + \frac{C}{x^2}$ $\quad y(1) = -1 + C = 0$

$\qquad\qquad\qquad\qquad\quad +1 \qquad +1$

$\boxed{C=1}$

$\boxed{y = \frac{-1}{x} + \frac{1}{x^2}}$

Checking Work: $y' = \frac{1}{x^2} - \frac{2}{x^3}$

$x^2\left(\frac{1}{x^2} - \frac{2}{x^3}\right) + 2x\left(\frac{-1}{x} + \frac{1}{x^2}\right) + 1$

$1 - \frac{2}{x} - 2 + \frac{2}{x} + 1 = 1 - 2 + 1 = 0$ ✓

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2. (10 points) Solve the homogeneous equation:

$$(3x + 2y)dx + xdy = 0.$$

$3tx + 2ty = t(3x+2y)$

$tx = t(x)$

Homogenous at degree 1

$y = xv$

$dy = vdx + xdv$

$v = \frac{y}{x}$

$(3x + 2xv)dx + x(vdx + xdv) = 0$

$\downarrow$

$(3x + 2xv + xv)dx + x^2 dv = 0 \Rightarrow (3x + 3xv)dx + x^2 dv = 0$

$\frac{3x(1+v)\,dx}{x^2(1+v)} + \frac{x^2 dv}{x^2(1+v)} = 0$

$\Downarrow$

$\int \frac{3dx}{x} + \int \frac{dv}{1+v} = \int 0$

$\Downarrow$

$3\ln|x| + \ln|1+v| = k$

$\Downarrow$

$e^{\ln|x^3(1+v)|} = e^k \Rightarrow |x^3(1+v)| = e^k \Rightarrow x^3(1+v) = \pm e^h$

$\downarrow$

$\frac{x^3(1+v)}{x^3} = \frac{C}{x^3}$

$1 + v = \frac{C}{x^3} \Rightarrow 1 + \frac{y}{x} = \frac{C}{x^3} - 1$
-1

$x \cdot \frac{y}{x} = \left(\frac{C}{x^3} - 1\right)x$ $\boxed{y = \frac{C}{x^2} - x}$

3. Consider the following differential equation:

$$(5x^3 + 2y^2)dx + 2yxdy = 0$$

(a) (5 points) The above differential equation has a one-variable integrating factor (i.e. $\mu(x)$ or $\mu(y)$) Find the intergrating factor.

$P = 5x^3 + 2y^2$ $\quad \frac{\partial P}{\partial y} = 4y$

$Q = 2yx$ $\quad \frac{\partial Q}{\partial x} = 2y$

$h(x) = \frac{1}{2yx}(4y - 2y) = \frac{2y}{2yx} = \frac{1}{x}$ $\qquad g(y) = \frac{1}{5x^3+2y^2}(4y-2y) = \cancel{\frac{2y}{5x^3+2y^2}}$

$e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$

Integrating Factor: $\boxed{x}$

(b) (10 points) Find the general solutions to the above differential equations.

$(5x^4 + 2y^2x)dx + 2yx^2dy = 0$ $\qquad \frac{d}{dy}(5x^4 + 2y^2x) = 4yx$ $\quad \frac{d}{dx}(2yx^2) = 4yx$

Exact Now!

$\int 2yx^2 dy = y^2x^2$

$\frac{d}{dx}(y^2x^2 + \phi(x)) = 2y^2x + \phi'(x) = 2y^2x + 5x^4$

$\phi'(x) = 5x^4$ $\qquad \phi(x) = x^5$

General Solution: $\boxed{F(x,y) = x^5 + x^2y^2 = C}$

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4. Consider the autonomous equation:

$$y' = (y-1)(y-2)$$

(a) (5 points) Find the general solutions $y(t)$ to the above differential equations.

$y' = 0 \quad (y-1)(y-2) = 0 \Rightarrow y-1=0,\; y-2=0 \qquad \boxed{y \equiv 1 \\ y \equiv 2}$

They are solutions

$y' = y^2 - 3y + 2 \Rightarrow \int \frac{y'}{(y-1)(y-2)} = \int 1\, dt$

$\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}$

$1 = A(y-2) + B(y-1)$

Let $y=2$ $\quad 1 = B(1) \quad B = 1$

Let $y=1$ $\quad 1 = A(-1) \quad A = -1$

$\int \left(\frac{-1}{y-1} + \frac{1}{y-2}\right) dy = t + c$

$e^{\ln|y-2| - \ln|y-1|} = e^{t+c} \Rightarrow \left|\frac{y-2}{y-1}\right| = e^c e^t$

$\frac{y-2}{y-1} = ce^t$

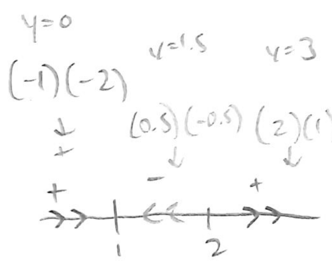
$y - 2 = yce^t - ce^t$

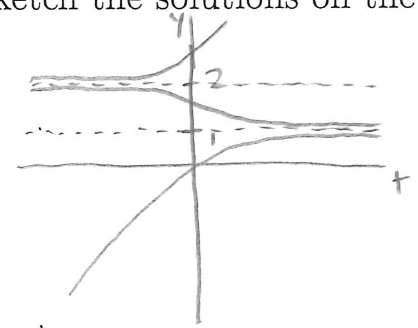
$y - yce^t = 2 - ce^t$

$\frac{y(1-ce^t)}{1-ce^t} = \frac{2-ce^t}{1-ce^t}$

$\boxed{y = \frac{2-ce^t}{1-ce^t}}$

(b) (3 points) Sketch the solutions on the $t-y$ plane.





(c) (4 points) Prove that if $y(t)$ is a solution and $y(0) = 1.9$, then $1 < y(t) < 2$ for all $t \in (-\infty, \infty)$

Let $y' = f(t,y) = y^2 - 3y + 2$

Then $f'(t,y) = 2y - 3$

$2y-3$ is continuous everywhere for all $t \in (-\infty, \infty)$. So uniqueness theorem applies.

Since $1 < 1.9 < 2$ and $y(t)$ cannot cross $y \equiv 1$ and $y \equiv 2$, $y(t)$ is between 1 and 2 for all $t \in (-\infty, \infty)$. ∎

(d) (3 points) Let $y(t)$ be the funtion in part(c). Calculate $\lim_{t\to+\infty} y(t)$.

Using the solution in part c and the sketch in part b,

$\lim_{t \to +\infty} y(t) = 1$

Scratch Paper

Checking work $y = \frac{2 - ce^t}{1 - ce^t}$ $y(1 - ce^t) = 2 - ce^t$

$$y' = \frac{-ce^t(1-ce^t) - ce^t(2-ce^t)}{(1-ce^t)^2}$$

$$= \frac{-ce^t}{1-ce^t} + \frac{ce^{2t} - 2ce^t}{(1-ce^t)^2}$$

$$\searrow \frac{ce^{2t} - 2ce^t}{ce^{2t} - 2ce^t + 1}$$

$$y^2 - 3y + 2$$

$$-ce^t(1-ce^t)$$

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Some useful formulas, etc:

Integrating factor $u(x)$ of a 1st Order Linear DE $x' = ax + f$:

$$u(x) = e^{-\int a(t)dt}$$

Single variable integrating factor μ for $Pdx + Qdy = 0$

- If $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,

$$\mu(x) = e^{\int h(x)dx}$$

- If $g(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,

$$\mu(y) = e^{-\int g(y)dy}$$