20W-MATH33B-1 Midterm 1

HENRY MACARTHUR

TOTAL POINTS

49 / 50

QUESTION 1

1 Q1 10 / 10

✓ **- 0 pts** Correct

- 3 pts Find the integrating factor or homogeneous solution.

- 2 pts Multiply whole equation by integrating factor

- 2 pts Integrate the equation

- 3 pts Solve the initial value problem

- 1 pts Minor computational mistake

- 2 pts Wrong sign in integrating factor/homogeneous solution

- 2 pts Find v' in variation of parameters

QUESTION 2

2 Q2 10 / 10

✓ **- 0 pts** Correct

- 0 pts Solve explicitly for y.

- 2 pts Algebra/integration error (see explanation/arrow)

- 1 pts Solutions are given by setting F(x,y) = C; you've just written out F(x,y).

- 1 pts Simplify by clearing out the natural logs and/or absolute values.

- 2 pts Where is your undetermined constant C?

- 1 pts Rewrite v in terms of y and x.

QUESTION 3

Q3 15 pts

3.1 (a) 5 / 5

✓ **- 0 pts** Correct

- 2 pts correct idea

- 3 pts no explanation but right answer

- 1 pts miscellaneous mistakes

- 4 pts tried

3.2 (b) 10 / 10

✓ **- 0 pts** Correct

- 1 pts miscellaneous mistake

- 8 pts tried

- 6 pts used exactness

- 5 pts had the right idea

- 3 pts had the right idea and made a logical mistake

QUESTION 4

15 pts

4.1 a 5 / 5

+ 5 **Point adjustment**

4.2 b 2.5 / 3

- 0 pts Correct

- 1 pts forget y<1 and y >2

- 3 pts blank answer

- 1 pts solutions can not cross each other

- 0.5 pts not dotted line. solution curve is continous

- 1 pts picture not correct.

- 2 pts where is the solution curves

✓ **- 0.5 pts** **0<y<1 should be S-shape**

- 2.5 pts not correct

- 1 pts 0<y<1 not correct

4.3 c 3.5 / 4

- 0 pts Correct

- 4 pts blank

- 3 pts not a proof.

- 1 pts didn't check \partial f / \ partial y

✓ **- 0.5 pts** **\partial f / \ partial y wrong**

- 2 pts Wrong theorem conditions.

- 2 pts didn't check theorem condition

- 1 pts didn't calculate partial derivative

- **- 1 pts** More detail
- **- 0.5 pts** minor mistake
- **- 0 pts** correct

4.4 d 3 / 3

✓ **- 0 pts** Correct
- **- 3 pts** not correct
- **- 2 pts** with some reason
- **- 1 pts** right track, wrong answer

Midterm 1

Last Name: MacArthur

First Name: Henry

Student ID: 705096169

Signature:

Section:

Tuesday:	Thursday:	
[✓ 1A]	1B	TA: YIH, SAMUEL
1C	1D	TA: KIM, BOHYUN
1E	1F	TA: BOSCHERT, NICHOLAS

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you.. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please **circle or box your final answers.**

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
Total:	50	

1. (10 points) Solve the initial value problem:

$$x^2y' + 2xy + 1 = 0, y(1) = 0$$

$x^2y' + 2xy + 1 = 0$, $y(1) = 0$

$x^2y' + 2xy + 1 = 0$

$x^2y' = -2xy - 1$

$y' = \frac{-2xy}{x^2} - \frac{1}{x^2}$

$y' = -\frac{2y}{x} - \frac{1}{x^2}$

$\mu = e^{-\int -\frac{2}{x}dx}$

$= e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = e^{\ln(x^2)}$

$= x^2$

$[\mu y]' = (-\frac{1}{x^2})\mu$

$[x^2y]' = (\frac{-1}{x^2})(x^2)$

$\int [x^2y]' = \int -1\,dx$

$x^2y = -x + c$

$y(1) = 0$

solve IVP

$x^2y + x = c$

$\cancel{(1)(0)}^{0} + (1) = c$

$c = 1$

$x^2y + x = 1$

$x^2y = 1 - x$

$\boxed{y = \frac{1-x}{x^2}}$

IOE: $(0, \infty)$

705

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2. (10 points) Solve the homogeneous equation:

$$(3x + 2y)dx + xdy = 0.$$

$(3tx + 2ty)dx + txdy = 0$

$t(3x+2y)dx + txdy = 0$ 1^{st} order

$\underline{y = xv}$

$dy = x\,dv + v\,dx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$(3x + 2y)dx + xdy = 0$

$xdy = -(3x+2y)dx$

$\frac{xdy}{dx} = -(3x+2y)$

$\frac{dy}{dx} = \frac{-3x - 2y}{x}$ $y = xv$

$x\frac{dv}{dx} + v = \frac{-3x - 2(xv)}{x} = \frac{\cancel{x}(-3-2v)}{\cancel{x}} = -3-2v$

$x\frac{dv}{dx} + v = -3 - 2v$

$x\frac{dv}{dx} = -3 - 3v$

$\frac{x}{dx} = \frac{-3-3v}{dv}$

$\frac{dx}{x} = \frac{dv}{-3-3v}$

$\int \frac{dx}{x} = \int \frac{dv}{-3(1+v)}$

$\ln|x| = -\frac{1}{3}\int \frac{dv}{1+v}$

$\ln|x| = -\frac{1}{3}\ln|1+v| + c$

let $A = e^c$ $x = e^{-\frac{1}{3}\ln|1+v| + c}$

$x = e^c e^{-1/3 \ln|1+v|}$

$x = A \cancel{e^{\ln}}(1+v)^{-1/3}$

$x = A\frac{1}{(1+v)^{1/3}}$

$y = xv$

$v = \frac{y}{x}$

$$\boxed{x = A\left(\frac{1}{1+y/x}\right)^{1/3}}$$

$B = A^3$

$x(1+\frac{y}{x})^{1/3} = A$

$x^3(1+\frac{y}{x}) = B$

$x^3 + \frac{x^3y}{x} = B$

$x^3 + x^2y = B$

$x^2y = B - x^3$

$y = \frac{B - x^3}{x^2}$

$2yx^2$

$5x^4 + 2xy^2$ $\quad$ $4xy$

$4xy$

3. Consider the following differential equation:

$$\overset{P}{(5x^3 + 2y^2)dx} + \overset{Q}{2yxdy} = 0$$

(a) (5 points) The above differential equation has a one-variable integrating factor (i.e. $\mu(x)$ or $\mu(y)$) Find the intergrating factor.

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{2yx}(4y - 2y)$$

$$= \frac{1}{2yx}(2y) = \frac{\cancel{2y}}{\cancel{2y}x} = \frac{1}{x} = a$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$

$$\boxed{\mu(x) = x}$$

$\rightarrow$ $x(5x^3 + 2y^2)dx + x \cdot 2yxdy = 0$

$(5x^4 + 2y^2x)dx + 2yx^2dy = 0$

$\frac{\partial P}{\partial y} = 4yx$ $\qquad$ $\frac{\partial Q}{\partial x} = 4 \cdot yx$ $\quad$ ✓ exact

(b) (10 points) Find the general solutions to the above differential equations.

~~$\int 2yx^2 dy$~~
~~$y^2x^2 + \varphi(x)$~~
~~$2y^2x + \varphi'(x) =$~~

$$\underset{P}{\mu(5x^3 + 2y^2)dx} + \underset{Q}{\mu 2yxdy} =$$

$$(5x^4 + 2y^2x)dx + 2yx^2dy = 0$$

$$\int P dx = \int (5x^4 + 2y^2x)dx = x^5 + y^2x^2 + \varphi(y)$$

$$\frac{\partial}{\partial y}(x^5 + y^2x^2 + \varphi(y)) = 2yx^2 + \varphi(y) = \frac{\partial F}{\partial y}$$

$$\cancel{2yx^2} + \varphi'(y) = \cancel{2yx^2}$$

$\int \varphi'(y) \neq 0$

$\varphi'(y) = c$

$$\boxed{F(x,y) = x^5 + y^2x^2 = C}$$

$y^2x^2 = C - x^5$

$y^2 = C - x^5 - x2$

$y = \pm\sqrt{C - x^5 - x2}$

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$(y-1)(y-2)$

$y^2 - 2y - y + 2$

$y - 3y + 2$

4. Consider the autonomous equation:

$$y' = (y-1)(y-2)$$

(a) (5 points) Find the general solutions $y(t)$ to the above differential equations. $y=1, y=2$

$\frac{dy}{dx} = (y-1)(y-2)$

$y' = (y-1)(y-2)$ $\int \frac{dy}{(y-1)(y-2)} = \int dx$

$A = e^c$

$\frac{A}{y-1} + \frac{B}{y-2} = 1$

$A(y-2) + B(y-1) = 1$

$B(2-1) = 1$

$B = 1$

$A(1-2) + 0 = 1$

$A(-1) = 1$

$A = -1$

$\int \frac{-1}{y-1} dy + \int \frac{1}{y-2} dy = \int dx$

$-\ln|y-1| + \ln|y-2| = x + c$

$\ln|y-2| - \ln|y-1| = x + c$

$\ln\left|\frac{y-2}{y-1}\right| = x + c$

$\frac{y-2}{y-1} = e^{x+c}$

$\frac{y-2}{y-1} = Ae^x$

$y-2 = (y-1)(Ae^x)$

$y-2 = Ae^x y - Ae^x$

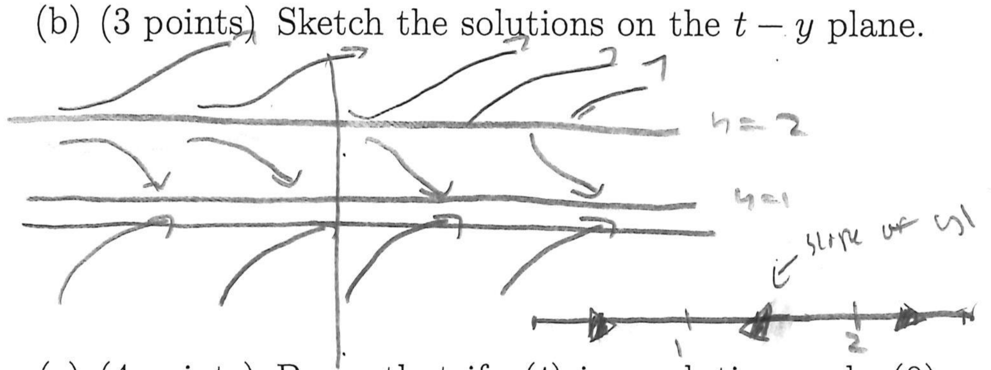
$y = Ae^x y - Ae^x + 2$

$y - Ae^x y = -Ae^x + 2$

$y(1-Ae^x) = -Ae^x + 2$

$$\boxed{y = \frac{-2-Ae^x}{1-Ae^x}}$$

(b) (3 points) Sketch the solutions on the $t - y$ plane.



$\frac{3}{2}$ 3/2

$y' = (y-1)(y-2)$

(c) (4 points) Prove that if $y(t)$ is a solution and $y(0) = 1.9$, then $1 < y(t) < 2$ for all $t \in (-\infty, \infty)$

since y and y' are continuous on $(-\infty, \infty)$ let R (rect) be the entire t-y plane, so $(0, 1.9) \in R$. by E thm since y exists on R and R contains $(0, 1.9)$. There exists a sol to y' s.t. $y(0) = 1.9$. using $y' = (y-1)(y-2)$ we can take $y(t) = 1$ and $y(t) = 2$ to be two general solutions. since y' is continuous on R, and $\frac{\partial f}{\partial y} = y - 3y + 2$, which is also cont on R,

let $f = (y-1)(y-2)$ where $y' = f(y)$

(d) (3 points) Let $y(t)$ be the funtion in part(c). Calculate $\lim_{t\to+\infty} y(t)$.

the Uniq thm states that while $y(t) = 1.9$ $y_1(t) = 1$ $y_2(t) = 2$, and $g(0) = 1.9$ $y_1(0) = 1$ $y_2(t) = 2$, since $y(t) \neq y_1(t) \neq y_2(t)$ we know they are each uniq solutions that do not intersect

$y(t), y_1(t), y_2(t)$

since $(y_1(0) = 1) < (y(0) = 1.9) < (y_2(0) = 2)$ we know for all $t \in (-\infty, \infty)$, $1 < y(t) < 2$ ☑

on back

part d →

4 d)

goes to ∞

$$\lim_{t\to+\infty} y(t) = \frac{2-Aex}{1-Aex} = \frac{2-\infty}{1-\infty} = \underline{1}$$

goes to ∞

$$\boxed{\lim_{t\to+\infty} y(t) = \underline{1}}$$

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4 a) $y' = (y-1)(y-2)$

$$\frac{dy}{(y-1)(y-2)} = dx$$

$$\frac{A}{(y-1)} + \frac{B}{(y-2)} = 1 \qquad \underline{=1}$$

$A(y-2) + B(y-1) = 1$

$A(-1) = 1$

$A = -1 \qquad B = 1$

Scratch Paper

$y = xv$

$dy = x\,dv + v\,dx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$$(3x + 2y)dx + x\,dy = 0$$

$$x\,dy = -(3x+2y)dx$$

$$\frac{dy}{dx} = \frac{-3x-2y}{x}$$

$$x\frac{dv}{dx} + v = \frac{-3x-2xv}{x}$$

$$x\frac{dv}{dx} + v = -3-2v$$

$$x\,dv/dx = -3-3v$$

$$\int\frac{dx}{x} = \int\frac{dv}{-3-3v}$$

$$\ln|x| = \int\frac{dv}{-3(1+v)}dv$$

$$\ln|x| = -\frac{1}{3}\int\frac{dv}{1+v}dv$$

$$\ln|x| = -\frac{1}{3}\ln|1+v| + C$$

$$x = Ae^{-1/3\ln|1+v|}$$

$$x = A\cancel{e^{\ln}}(1+v)^{-1/3}$$

$$x = A/(1+v)^{+1/3}$$

$$x = \frac{A}{(1+y/x)^{1/3}}$$

$$x(1+y/x)^{1/3} = A$$

$$x^3(1+y/x) = A^3$$

$$x^3 + \frac{yx^3}{x} = A^3$$

$$x^3 + yx^2 = A$$

$$yx^2 = A - x3$$

$$y = \frac{A-x^3}{x^2}$$

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Some useful formulas, etc:

Integrating factor $u(x)$ of a 1st Order Linear DE $x' = ax + f$:

$$u(x) = e^{-\int a(t)dt}$$

Single variable integrating factor μ for $Pdx + Qdy = 0$

- If $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,

$$\mu(x) = e^{\int h(x)dx}$$

- If $g(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,

$$\mu(y) = e^{-\int g(y)dy}$$