1. (a) (5 points) Find the solution y_h to the differential equation:

$$y' = \frac{1}{x} y$$

$y' = a(x) y$

according to formulas in the textbook

$$y_h = Ce^{\int a(x)dx}$$
$$= Ce^{\int \frac{1}{x}dx}$$
$$= Ce^{\ln|x|}$$
$$= C|x|$$
$$= Cx, \text{ where } C \in (-\infty, \infty)$$

(b) (10 points) Solve the initial value problem:

$$y' = \frac{1}{x} y + \sqrt{x}, y(1) = 0$$

using variation of parameter

pick up $y_{h1} = x$, let $C = 1$ → $y' = a(x) y + f(x)$.

$y = v y_{h1}$ since $v' = \frac{f}{y_{h1}}$ (proved in textbook).

$$v' = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$v = \int x^{-\frac{1}{2}} dx = 2\sqrt{x} + C$$

$$\therefore y = v \cdot y_{h1} = (2\sqrt{x} + C) \cdot x = 2x^{\frac{3}{2}} + Cx$$

plug $y(1) = 2 \cdot 1^{\frac{3}{2}} + C = 0$

$$C = -2$$

$$\therefore y = 2x^{\frac{3}{2}} - 2x$$

Scanned with CamScanner

Last six digits of UID: 206804

2. (a) (5 points) Find the general solution $y_h = C_1y_1 + C_2y_2$ to the differential equation:

$$y'' + y = 0$$

$\Rightarrow \lambda^2+1=0 \quad \lambda^2=-1 \quad \lambda=\pm\sqrt{-1}$

$\therefore$ complex roots solution $\pm i \quad \therefore a=0 \quad b=1$

$\therefore y_1 = \sin t \quad y_2 = \cos t$

$y_h = C_1 \sin t + C_2 \cos t$

(b) (10 points) Use undetermined coefficient or variation of parameters, find the general solution to the differential equations

$$y'' + y = t + e^t$$

Solve $y''+y=t$ and $y''+y=e^t$ respectively

Let $y_{p1} = at+b$

$y_{p1}' = a \quad y_{p1}''=0$

plug back: $0+at+b=t$

$b=0 \quad a=1$

$\therefore y_{p1} = t$

Let $y_{p2} = ke^t$

$y_{p2}' = ke^t \quad y_{p2}'' = ke^t$

plug back $ke^t+ke^t=e^t$

$2ke^t = e^t$

$k=\frac{1}{2}$

$\therefore y_{p2} = \frac{1}{2}e^t$

by theorem on linearity $y_p = y_{p1} + y_{p2}$ is a particular solution to $y''+y=t+e^t$

$\therefore y_p = t+\frac{1}{2}e^t$

$\therefore$ general solution $= y_h + y_p = C_1\sin t + C_2\cos t + t + \frac{1}{2}e^t$

3. (10 points) Solve the homogeneous equation:

$$(y^2 + 2xy)dx - x^2dy = 0$$

(Hint: Using $y = vx$ change the differential equation to a seperable equation)

both y^2+2xy and $-x^2$ are homegeneous of degree 2

$\therefore$ Let $y = vx$ $dy = dv\,x + dx\,v$ and plug into the original equation we get

$(v^2x^2 + 2x^2v)dx - x^2(dv\,x + dx\,v) = 0$

divide x^2 from both sides.

$(v^2+2v)dx - dv\,x - dx\,v = 0.$

$(\underset{v^2+v}{v^2+2v-v})dx - dv\,x = 0$

$(v^2+v)dx = x\,dv$

$\frac{dx}{x} = \frac{dv}{v^2+v}$

integrate both sides we get

$\int \frac{dx}{x} = \int \frac{dv}{v(v+1)}$

-5

$\ln|x| = \cancel{\frac{\ln|v^2+v|}{2v+1}} + C$

use e to equation then plug $y = vx$ back we see

$e^{\ln|x|} = e^{\frac{\ln|v^2+v|}{2v+1}} \cdot e^C$

$|x| = C|v^2+v|^{2v+1}$

$|x| = C\left|\frac{y^2}{x^2} + \frac{y}{x}\right|^{2\frac{y}{x}+1}$

Last six digits of UID: 206804

4. Consider the autonomous equation:

$$y' = y(y-2)e^y = (y^2-2y)e^y = y^2e^y - 2ye^y$$

(a) (2 points) Find the equilibrium solutions of the above differential equations.

$0 = C(C-2)e^C$

$C_1 = 0 \quad C_2 = 2$

$\therefore y_1 = 0$ and $y_2 = 2$ are two equil. solutions.

(b) (3 points) Determine the stability of the equilibrium solutions.

(test $y=-1$)

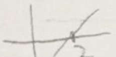
for $y<0$, $y' = (-1)(-1-2)e^{-1}$ is positive since e^c is always positive

test $y=1$

$2>y>0$ $\quad y' = (1)(1-2)e^1$ is negative

$\therefore y_1 = 0$ is stable ($y''<0 \to$ stable, theorem intuitively)

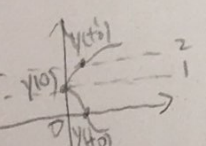
for $y_2 = 2$ $\quad 2>y>0$ y' is negative (shown).



$y>2$ $y=3$ $(3-2)e^y$ is positive $\therefore y_2=2$ is unstable ($y''>0$)

(c) (5 points) Prove that if $y(t)$ is a solution and $y(0) = 1$, then $0 < y(t) < 2$ for all $t \in (-\infty, \infty)$.

Since both $y_1=0$ and $y_2=2$ are equilibrium solutions and that $y' = y(y-2)e^y = f$ and $\frac{\partial f}{\partial y} = (y^2e^y - 2ye^y)'$ are continuous for the ty plane, by uniqueness and existence (U/E) theorem that there exists a unique solution to each initial value problem



now: suppose $y(t_1) \leq 0$, then by intermediate value theorem (since $y(0)=1$) there must be a $t_0 \in (t, t_1)$ such that $y(t_0)=0$ however, then for the initial value problem $y' = y(y-2)e^y$ $y(t_0)=0$ there will be two distinct solutions (1) $y(t)$ and (2) $y_1(t)=0$ which contradicts the U/E theorem $\therefore y(t)>0$ ①

Similarly, suppose $y(t_1') \geq 2$, then by intermediate value theorem there exists a $t_0' \in (t', t_1')$ such that $y(t_0')=2$. then there will be two distinct solutions to the I.V.P. (1) $y(t)$ (2) and $y_2(t)=2$, which contradicts the U/E theorem

$\therefore y(t)<2$ ② $\quad \therefore$ considering ① ②, $0<y(t)<2$ for all $t \in (-\infty, \infty)$