1. (a) (5 points) Find the solution y_h to the differential equation:

$$y' = \frac{1}{x}y$$

$\frac{dy}{dx} = \frac{1}{x}y$

$y' - \frac{1}{x}y = 0$

$\mu = e^{\int -\frac{1}{x}} = \frac{1}{x}$

$\int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\frac{1}{x}y' - \frac{1}{x^2}y = 0$

$(\frac{1}{x}y)' = 0$

$\ln(y) = \ln(x) + C$

$y = Cx$

$e^{\ln(y)} = e^{\ln(x)+C}$

→ be careful abt absolute values -1pt.

$\boxed{y_h = y = Cx}$ ✓ where C is a constant

$y = Cx$ $\quad y' = C = \frac{Cx}{x} = C$ ✓

(b) (10 points) Solve the initial value problem:

$$y' = \frac{1}{x}y + \sqrt{x}, y(1) = 0$$

$y' - \frac{1}{x}y = \sqrt{x}$

$\mu = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)} = \frac{1}{x}$

$0 = 2(1)^{3/2} + C(1)$

$C = -2$

$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{\sqrt{x}}$

$\boxed{y = 2x^{3/2} - 2x}$

$\int (\frac{1}{x}y)' dy = \int \frac{1}{\sqrt{x}}dx$

$3x - 2 = \frac{1}{x}(2x^{3/2} - 2x) + \sqrt{x}$

$\frac{1}{x}y = 2\sqrt{x} + C$

$3x - 2 = 2x^{1/2} - 2 + \sqrt{x}$

$3x - 2 = 3x - 2$ ✓

$y = 2x^{3/2} + Cx$

$y(1) = 0$

2. (a) (5 points) Find the general solution $y_h = C_1 y_1 + C_2 y_2$ to the differential equation:

$$y'' + y = 0$$

$y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

so Y_h is in the form $\boxed{Y_h = C_1 \cos(t) + C_2 \sin(t)}$ ✓

(b) (10 points) Use undetermined coefficient or variation of parameters, find the general solution to the differential equations

$$y'' + y = t + e^t$$

using above

$Y = Y_h + Y_p$

$Y_h = C_1 \cos(t) + C_2 \sin(t)$

split up particular

$Y_p = at + b$ $\quad Y_p'' + Y_p = t$ $\quad Y_p = ae^t$ $\quad Y_p'' + Y_p = e^t$

$Y_p' = a$ $\quad 0 + at + b = t$ $\quad Y_p' = ae^t$ $\quad ae^t + ae^t = e^t$

$Y_p'' = 0$ $\quad a = 1 \;\; b = 0$ $\quad Y_p'' = ae^t$ $\quad 2ae^t = e^t$

$a = \frac{1}{2}$

$Y_p = t$ $\quad Y_p = \frac{1}{2}e^t$

recombine $\quad Y_p = t + \frac{1}{2}e^t$

general solution: $\quad Y = Y_h + Y_p$ ✓

$$\boxed{Y = C_1 \cos(t) + C_2 \sin(t) + t + \frac{1}{2}e^t}$$

$y'' + y = \frac{1}{2}e^t + \frac{1}{2}e^t + t = t + e^t$ ✓

3. (10 points) Solve the homogeneous equation:

$$(y^2 + 2xy)dx - x^2 dy = 0$$

(Hint: Using $y = vx$ change the differential equation to a seperable equation)

$(y^2+2xy)dx - x^2 dy = 0$

$P = y^2+2xy \quad P_y = 2y+2x$

$Q = -x^2 \quad Q_x = -2x \quad$ not exact!

let $y = vx$

$dy = v dx + x dv$

$(v^2x^2 + 2vx^2)\,dx - x^2 v dx - x^3 dv = 0$

$= (v^2x^2 + vx^2)\,dx - x^3 dv = 0$

$= \dfrac{(v^2x^2 + vx^2)dx}{x^2} = \dfrac{x^3 dv}{x^2}$

$= (v^2+v)\,dx = x dv \qquad \dfrac{A}{v} + \dfrac{B}{v+1}$

$\dfrac{1}{x} dx = \dfrac{1}{v^2+v} dv \qquad Av + A + Bv = 1$

$A = 1$

$\ln(x) = \int \frac{1}{v} - \frac{1}{v+1} \qquad B = -1$

$\ln(v) - \ln(v+1) + C$

$e^{\ln(x)} = e^{\ln(\frac{v}{v+1}) + C} \qquad v = \frac{y}{x}$

$x = \frac{y}{x+y}$

$x = C^*(\frac{v}{v+1})$

where $C^* = e^C$

$\boxed{x = C^*\left(\dfrac{\frac{y}{x}}{\frac{y}{x}+1}\right)}$ AND?

-1

$x = C^* \frac{y}{y+x}$

$\frac{x^2+xy}{y} = C^*$

Last six digits of UID: [redacted]

4. Consider the autonomous equation:

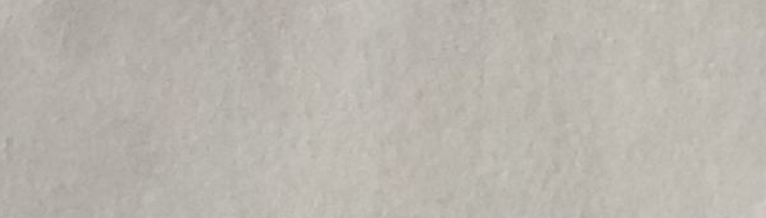
$$y' = y(y-2)e^y$$

(a) (2 points) Find the equilibrium solutions of the above differential equations. Equilibrium solutions ar $y=0, 2$

(b) (3 points) Determine the stability of the equilibrium solutions.

$f=(y^2-2y)e^y$

$f' = (2y-2)e^y + (y^2-2y)e^y = (y^2-2)e^y$



phase line confirms 0 stable 2 unstable

plugging into f'

$\boxed{0 \text{ is stable} \quad 2 \text{ is Unstable}}$

(c) (5 points) Prove that if $y(t)$ is a solution and $y(0) = 1$, then $0 < y(t) < 2$ for all $t \in (-\infty, \infty)$.

1. Let $q(t)=0$ and $g(t)=2$ and both are valid solution, for $(-\infty,\infty)$ to the given ODE

2. as $f=(y^2-2y)e^y$ and $f'=(y^2-2)e^y$ are both continuous the differential follows uniqueness meaning no two solutions may ever cross

3. if $y(t)$ were <0 or >2 for any t, to be 1 at $t=0$ it would have to cross the solutions $g(t)=2$ or $q(t)=0$ at some point according to the IVT

4. Therefore it must be that $0 < y(t) < 2$ for $t \in (-\infty, \infty)$