20W-MATH33B-1 Final Exam

JONATHAN CHAU

TOTAL POINTS

95 / 100

QUESTION 1

1 Question 1 10 / 10

✓ **- 0 pts** Correct

- 1 pts minus sign error

- 2 pts need to be in the form of "y = " form

- 3 pts algebraic mistake need to be ln(x)

- 1 pts miscelleneous algebraic mistake

- 5 pts only the homogeneous equation substitution was right

- 2 pts the position of "+c" needs to be in denominator

- 8 pts tried

QUESTION 2

2 Question 2 15 / 15

✓ **- 0 pts** Correct

- 3 pts some portion particular solution wrong/not found correctly

- 1 pts miscellaneous mistake

- 6 pts answer for part (b) wrong

- 2 pts general solution for 2(a) incomplete

QUESTION 3

Question 3 15 pts

3.1 3 (a) 10 / 10

✓ **- 0 pts** Correct

- 4 pts Miscomputed one eigenvector.

- 2 pts You found fundamental solutions, but forgot C_1 and C_2.

- 3 pts Miscomputed both eigenvectors.

- 1 pts Write down the general solution, not just fundamental solutions.

3.2 3(b) 5 / 5

✓ **- 0 pts** Correct

- 1 pts Justification?

- 2 pts Eigenvectors graphed in incorrect quadrants.

- 2 pts Indicate direction travelled on solution curves.

- 2 pts The shape of your curves as t goes to infinity or - infinity is wrong

- 3 pts Draw solution curves in quadrants cut out by eigenvectors.

- 3 pts Draw the half-line solutions (the ones corresponding to the eigenvectors).

QUESTION 4

4 Question 4 10 / 10

✓ **- 0 pts** Correct

- 2 pts Your third fundamental solution is wrong/missing.

- 2 pts Your second fundamental solution is wrong/missing.

- 1 pts e^{2t} not e^t.

- 2 pts System of coefficients solved incorrectly/solution to IVP missing/incorrect.

- 2 pts Your first fundamental solution is wrong/missing.

QUESTION 5

5 Question 5 15 / 15

✓ **- 0 pts** Correct

- 2 pts Identify block matrices

- 3 pts Find eigenvalues for each block

- 3 pts Find (generalized) eigenvectors for each block

- 4 pts Construct solutions for each block

- 3 pts Combine solutions.

- 1 pts Minor calculation error

- 2 pts Moderate error in solution for one block

QUESTION 6

Question 6 15 pts

6.1 6(a) **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts Knew to find zeros of RHS.

- 2 pts Correctly found at least one infinite family of solutions.

- 1 pts Found half of the solutions or made a computational mistake.

6.2 6(b) **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts Included equilibria

- 3 pts Solutions go in correct directions

- 1 pts Violates uniqueness

- 1 pts Small error

6.3 6(c) **4 / 5**

- 0 pts Correct

- 2 pts Invoke hypotheses of existence and uniqueness.

- 3 pts Bound by equilibria

- 1 pts Minor error

- 1 pts Show it satisfies the hypotheses (show f and f' are continuous).

✓ **- 1 pts** **State that there are arbitrarily large or small equilibria.**

- 1 pts Invoke existence and uniqueness.

QUESTION 7

Question 7 10 pts

7.1 7(a) **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts y_h /IF correct, didn't find y_p

- 2 pts minor mistake / gap

- 4 pts Major mistake/gap

- 5 pts blank

- 3 pts y_h/IF minor mistake/not simplified, didn't find y_p

- 1 pts y_h not simplifed

- 3 pts y_h = ?

- 1 pts typo

7.2 7(b) **3 / 3**

✓ **- 0 pts** **Correct:**

- 1 pts minor mistake (e.g. forget to say C can be anything), gap, logic flow not clear

- 2 pts some meaningful writings. not much detail provided, many gaps.

- 3 pts nothing meaningful

7.3 7(c) **0 / 2**

- 0 pts Correct

- 2 pts wrong

✓ **- 1 pts** **didn't put in normal form y' = F(y,t) = 1/ (t^2 - a^2) y + ...**

✓ **- 1 pts** **didn't check/state that \partial F/ \partial y = 1/(t^2 - a^2) or calculation is wrong**

- 1 pts Gap

QUESTION 8

8 Question 8 **8 / 10**

- 0 pts Correct

- 4 pts gap: did not verify (A-aI)^2 = 0

- 4 pts minor mistake

✓ **- 2 pts** **lack essential detail / some typos**

- 4 pts based on your flow, you didn't use math induction to give a proof for A^n

- 8 pts Major mistake

1. (10 points) Solve the homogeneous equation (Your final answer should be in $y = f(x, C)$ form, e.g $y = \frac{1}{C+x}$):

$$(-xy + y^2)dx + x^2dy = 0.$$

$-xy\,dx + y^2dx + x^2dy = 0$

$y = vx \qquad v = \frac{y}{x}$

$dy = v\,dx + x\,dv$

$\downarrow$

$-x^2v\,dx + v^2x^2dx + x^2v\,dx + x^3dv$

$\downarrow$

$\frac{v^2x^2\,dx}{x^3v^2} + \frac{x^3dv}{x^2v^2} = 0$

$\frac{dx}{x} + \frac{dv}{v^2} = 0 \Rightarrow \int\frac{dx}{x} = \int\frac{-dv}{v^2}$

$\downarrow$

$\ln|x| + C = \frac{1}{v}$

$\downarrow$

$\frac{y}{\ln|x|+C}\cdot(\ln|x|+C) = \frac{x}{y}\cdot\frac{y}{\ln|x|+C}$

$$\boxed{y = \frac{x}{\ln|x|+C}}$$

1 Question 1 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** minus sign error
- **- 2 pts** need to be in the form of "y = " form
- **- 3 pts** algebraic mistake need to be ln(x)
- **- 1 pts** miscelleneous algebraic mistake
- **- 5 pts** only the homogeneous equation substitution was right
- **- 2 pts** the position of "+c" needs to be in denominator
- **- 8 pts** tried

Last six digits of UID: 166 732

2. (a) (5 points) Find the general solution to the differential equation:

$$y'' - 2y' + y = 0$$

$r^2 - 2r + 1 = 0$

$(r-1)^2 \quad r = 1$

$y_1 = e^t$

$y_2 = te^{rt}$

↑ because of same root

$y = c_1y_1 + c_2y_2 \rightarrow$ $\boxed{y = c_1e^t + c_2te^t}$

(b) (10 points) Find a particular solution to the differential equation (Hint: split forcing term into two parts, check the table in P172 of your textbook):

$$y'' - 2y' + y = e^t(t+1) + e^t \sin t.$$

$te^t + e^t$

$t^2e(at+b) = at^3e^t + bt^2e^t$

$e^t(a\cos(t) + b\sin(t))$

$y' = 3at^2e^t + at^3e^t + 2bte^t + bt^2e^t$

$y'' = 6ate^t + 6at^2e^t + at^3e^t + 2be^t + 4bte^t + bt^2e^t$

$y'' - 2y' + y = e^t(t+1)$

$te^t + e^t$

$6ate^t + 6at^2e^t + at^3e^t + 2be^t + 4bte^t + bt^2e^t$

$- 6at^2e^t - 2at^3e^t \quad - 4bte^t - 2bt^2e^t$

$+ at^3e^t \quad + bt^2e^t$

$= 6ate^t + 2be^t$

$6ate^t = te^t \quad a = \frac{1}{6}$

$2be^t = e^t \quad b = \frac{1}{2}$

$y_{p1} = \frac{1}{6}t^3e^t + \frac{1}{2}t^2e^t$

$y' = a\cos(t)e^t - a\sin(t)e^t + b\sin(t)e^t + b\cos(t)e^t$

$y'' = a\cos(t)e^t - a\sin(t)e^t - a\sin(t)e^t - a\cos(t)e^t$
$+ b\sin(t)e^t + b\cos(t)e^t + b\cos(t)e^t - b\sin(t)e^t$

$= -2a\sin(t)e^t + 2b\cos(t)e^t$

$-2a\sin(t)e^t + 2b\cos(t)e^t$
$+ 2a\sin(t)e^t - 2b\cos(t)e^t - 2a\cos(t)e^t - 2b\sin(t)e^t$
$+ a\cos(t)e^t + b\sin(t)e^t$

$= -a\cos(t)e^t - b\sin(t)e^t$

$-a\cos(t)e^t = 0e^t\cos(t) \quad a = 0$

$-b\sin(t)e^t = e^t\sin(t) \quad b = -1$

$y_{p2} = e^t(-b\sin(t))$

$\boxed{y_p = \frac{1}{6}t^3e^t + \frac{1}{2}t^2e^t - e^t\sin(t)}$

2 Question 2 **15 / 15**

- ✓ **- 0 pts** Correct
- **- 3 pts** some portion particular solution wrong/not found correctly
- **- 1 pts** miscellaneous mistake
- **- 6 pts** answer for part (b) wrong
- **- 2 pts** general solution for 2(a) incomplete

3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

$\det(A - \lambda I) = \lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) \qquad \lambda = 2$

$\lambda = -1$

$\lambda = 2 \quad \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ row reduce $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

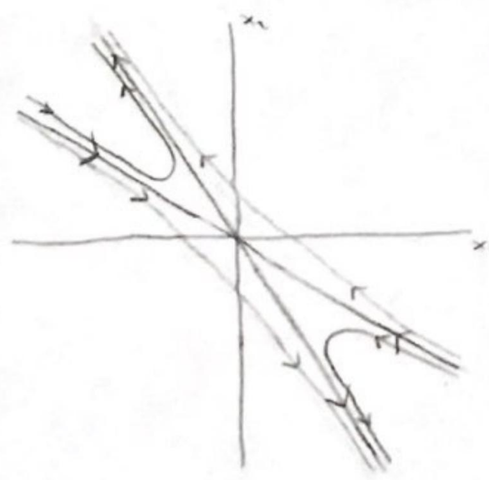
$V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\lambda = -1 \quad \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ row reduce. $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\boxed{y = C_1 e^{2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)



Saddle

3.1 3 (a) **10 / 10**

✓ **- 0 pts** Correct

- 4 pts Miscomputed one eigenvector.

- 2 pts You found fundamental solutions, but forgot C_1 and C_2.

- 3 pts Miscomputed both eigenvectors.

- 1 pts Write down the general solution, not just fundamental solutions.

3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

$\det(A - \lambda I) = \lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1)$ $\lambda = 2$

$\lambda = -1$

row reduce

$\lambda = 2$ $\begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

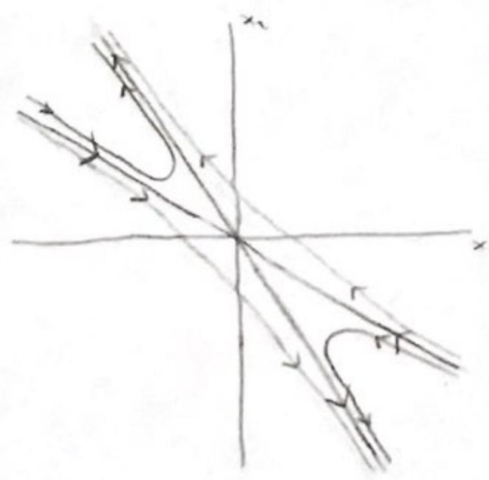
row reduce.

$\lambda = -1$ $\begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\boxed{y = C_1 e^{2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)



Saddle

3.2 3(b) **5 / 5**

✓ **- 0 pts** **Correct**

- 1 pts Justification?

- 2 pts Eigenvectors graphed in incorrect quadrants.

- 2 pts Indicate direction travelled on solution curves.

- 2 pts The shape of your curves as t goes to infinity or - infinity is wrong

- 3 pts Draw solution curves in quadrants cut out by eigenvectors.

- 3 pts Draw the half-line solutions (the ones corresponding to the eigenvectors).

Last six digits of UID: 166 732

4. (10 points) Find the solution $\mathbf{y}(t)$ to the following 3×3 system with given initial condition $\mathbf{y}(0) = (2,-2,1)^T$:

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial)

$$0 = (-1)^3 \det(A - \lambda I) = -\det\begin{pmatrix} 3-\lambda & 1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$= -\left((3-\lambda)(\lambda^2 - 3\lambda + 3) + (-1+2-\lambda) + (-1-1+\lambda)\right)$$

$$= -(3\lambda^2 - 9\lambda + 9 - \lambda^3 + 3\lambda^2 - 3\lambda + 1 - \lambda - 2 + \lambda)$$

$$= \lambda^3 - 6\lambda^2 + 12\lambda - 8$$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array} \quad (x-2)$$

$\pm 1, \pm 2, \pm 4, \pm 8$

$$(\lambda - 2)(\lambda - 4\lambda + 4) = (\lambda - 2)^3 \qquad \lambda = 2 \text{ alm: } 3$$

$$\lambda = 2 \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$(A - \lambda_1 I)v_2 = v_1 \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(A - \lambda I)v_3 = v_2 \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

$x_1 + x_2 = 1 \quad x_3 = -1$

$$y = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{2t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) + c_3 e^{2t}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t^2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right)$$

continued →

9) continued

$$y(0) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$c_1 + c_3 = 2$ $\quad$ $2 + c_3 = 2$ $\quad$ $c_3 = 0$

$-c_1 = -2$ $\quad$ $c_1 = 2$

$c_2 - c_3 = 1$ $\quad$ $c_2 = 1$

$$\boxed{y = 2e^{2t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + e^{2t} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)}$$

4 Question 4 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 2 pts** Your third fundamental solution is wrong/missing.
- **- 2 pts** Your second fundamental solution is wrong/missing.
- **- 1 pts** e^{2t} not e^t.
- **- 2 pts** System of coefficients solved incorrectly/solution to IVP missing/incorrect.
- **- 2 pts** Your first fundamental solution is wrong/missing.

5. (15 points) Find the general solution(fundamental set) $\mathbf{y}(t)$ to the following 6×6 system:

$$\mathbf{y}' = \left(\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right) \mathbf{y}$$

(Hint : This is a block matrix. Try find a 1 by 1, 3 by 3, and 2 by 2 block.)

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 & 0 & 0 \\ 0 & -1 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & -2 \\ 0 & 0 & 0 & 0 & 2 & -\lambda \end{pmatrix} = -\lambda(\lambda^2 - 2\lambda + 1)(1-\lambda)(\lambda^2 + 4)$$

$\lambda_1 = 0$

$\lambda_2 = 1$ alm: 3

$\lambda_3 = 2i$

$\lambda_4 = -2i$

$\frac{0 \pm \sqrt{0-4\cdot 4}}{2} \quad \frac{\pm 4i}{2}$

$$\lambda = 0 \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 1 \quad \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0 \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$(A - \lambda I)V_4 = V_3$ $\quad x_2 + x_3 = 1$, $-x_2 - x_3 = 1$; $x_2 = 1$, $x_3 = 0$

$$V_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 2i \quad \begin{pmatrix} -2i & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-2i & 1 & 0 & 0 & 0 \\ 0 & -1 & -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2i & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i & -2 \\ 0 & 0 & 0 & 0 & 2 & -2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0 \qquad V_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ i \\ 1 \end{pmatrix} \quad V_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -i \\ 1 \end{pmatrix}$$

$$y = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^t\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_4 e^t\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) + c_5\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_6\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix}$$

5 Question 5 **15 / 15**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Identify block matrices
- **- 3 pts** Find eigenvalues for each block
- **- 3 pts** Find (generalized) eigenvectors for each block
- **- 4 pts** Construct solutions for each block
- **- 3 pts** Combine solutions.
- **- 1 pts** Minor calculation error
- **- 2 pts** Moderate error in solution for one block

Last six digits of UID: 166 732

6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

Find $y' = 0$

$\sin y = -\cos y$

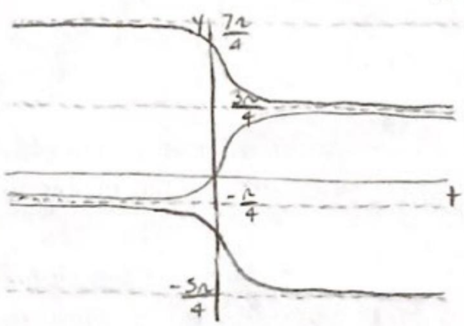
equilibria: $y = \frac{3\pi}{4} + n\pi$ n is integer in $(-\infty, \infty)$

$y = 0$

$y' = 1$

$y = \pi \to -1$

(b) (5 points) Sketch the solutions on the $t - y$ plane.



(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

Let $y' = f(t, y) = \sin y + \cos y$

Then $f'(t,y) = \cos y - \sin y$

$\cos y - \sin y$ is continuous everywhere for all $t \in (-\infty, \infty)$.

So uniqueness theorem applies.

Let $m = -\frac{\pi}{4}$ and $M = \frac{3\pi}{4}$

Due to the uniqueness theorem, $y(t)$ cannot cross and go below $m = -\frac{\pi}{4}$ and $y(t)$ cannot cross and go above $M = \frac{3\pi}{4}$.

Therefore, $y(t)$ is between m and M and is thus a bounded function. ∎

6.1 6(a) **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts Knew to find zeros of RHS.

- 2 pts Correctly found at least one infinite family of solutions.

- 1 pts Found half of the solutions or made a computational mistake.

Last six digits of UID: 166 732

6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

Find $y' = 0$

$\sin y = -\cos y$

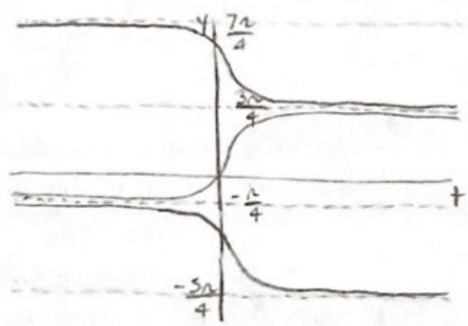
equilibria: $y = \frac{3\pi}{4} + n\pi$ n is integer in $(-\infty, \infty)$

$y=0$

$y' = 1$

$y = \pi \to -1$

(b) (5 points) Sketch the solutions on the $t - y$ plane.



(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

Let $y' = f(t, y) = \sin y + \cos y$

Then $f'(t,y) = \cos y - \sin y$

$\cos y - \sin y$ is continuous everywhere for all $t \in (-\infty, \infty)$.

So uniqueness theorem applies.

Let $m = -\frac{\pi}{4}$ and $M = \frac{3\pi}{4}$

Due to the uniqueness theorem, $y(t)$ cannot cross and go below $m = -\frac{\pi}{4}$ and $y(t)$ cannot cross and go above $M = \frac{3\pi}{4}$.

Therefore, $y(t)$ is between m and M and is thus a bounded function. ∎

6.2 6(b) **5 / 5**

- ✓ **- 0 pts** Correct
- **- 2 pts** Included equilibria
- **- 3 pts** Solutions go in correct directions
- **- 1 pts** Violates uniqueness
- **- 1 pts** Small error

Last six digits of UID: 166 732

6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

Find $y'=0$

$\sin y = -\cos y$

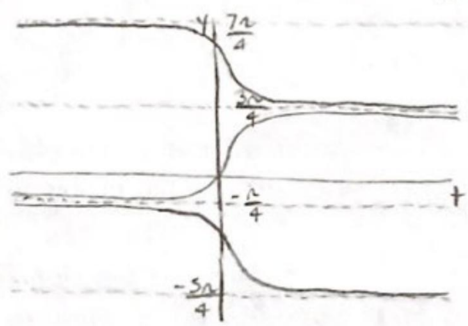
equilibria: $y = \frac{3\pi}{4} + n\pi$ n is integer in $(-\infty, \infty)$

$y=0$

$y'=1$

$y=\pi \to -1$

(b) (5 points) Sketch the solutions on the $t-y$ plane.



(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

Let $y' = f(t,y) = \sin y + \cos y$

Then $f'(t,y) = \cos y - \sin y$

$\cos y - \sin y$ is continuous everywhere for all $t \in (-\infty, \infty)$.

So uniqueness theorem applies.

Let $m = -\frac{\pi}{4}$ and $M = \frac{3\pi}{4}$

Due to the uniqueness theorem, $y(t)$ cannot cross and go below $m = -\frac{\pi}{4}$ and $y(t)$ cannot cross and go above $M = \frac{3\pi}{4}$.

Therefore, $y(t)$ is between m and M and is thus a bounded function. ∎

6.3 6(c) **4 / 5**

- **- 0 pts** Correct
- **- 2 pts** Invoke hypotheses of existence and uniqueness.
- **- 3 pts** Bound by equilibria
- **- 1 pts** Minor error
- **- 1 pts** Show it satisfies the hypotheses (show f and f' are continuous).
- ✓ **- 1 pts State that there are arbitrarily large or small equilibria.**
- **- 1 pts** Invoke existence and uniqueness.

7a) $y' = \frac{y+t^2-t-a^2}{t^2-a^2} = \frac{y-t}{t^2-a^2} + 1$

$y' - \frac{y}{t^2-a^2} = 1 - \frac{t}{t^2-a^2}$

Use variation of perameters $y = v y_h$

find y_h

$y' - \frac{y}{t^2-a^2} = 0 \Rightarrow \frac{1}{y} \cdot y' = \frac{y}{t^2-a^2} \cdot \frac{1}{y} \Rightarrow \int \frac{y'}{y} = \int \frac{1}{t^2-a^2} dt$

$+\frac{y}{t^2-a^2} \quad +\frac{y}{t^2-a^2}$

$e^{\ln(y_h)} = e^{\frac{\ln|t-a|}{2a} - \frac{\ln|t+a|}{2a}}$

$\int \frac{1}{t^2-a^2} dt \rightarrow \int \frac{1}{(t-a)(t+a)} dt \rightarrow \int \left(\frac{1}{2a(t-a)} - \frac{1}{2a(t+a)}\right) dt = \frac{1}{2a}\int \frac{1}{t-a} dt - \frac{1}{2a}\int \frac{1}{t+a} dt$

$\frac{1}{(t-a)(t+a)} = \frac{A}{(t-a)} + \frac{B}{(t+a)}$

$1 = A(t+a) + B(t-a)$

Let $t = a$

$\frac{1}{2a} = \frac{A \cdot 2a}{2a} \quad A = \frac{1}{2a}$

Let $t = -a$

$\frac{1}{-2a} = \frac{B(-2a)}{-2a} \quad B = \frac{1}{-2a}$

$u = t-a, \; du = dt \qquad u = t+a, \; du = dt$

$\frac{1}{2a}\int \frac{1}{u} du \qquad -\frac{1}{2a}\int \frac{1}{u} dt$

$\ln|u| \qquad \ln|u|$

$\frac{\ln|t-a|}{2a} - \frac{\ln|t+a|}{2a}$

$y_h = e^{\frac{\ln|t-a| - \ln|t+a|}{2a}}$

$y_h' = \frac{e^{\frac{\ln|t-a|-\ln|t+a|}{2a}}}{t^2-a^2} = \frac{y_h}{t^2-a^2}$

$y = v \cdot y_h$

$(vy_h)' = \frac{vy_h - t}{t^2-a^2} + 1 \Rightarrow v'y_h + vy_h' = \frac{vy_h}{t^2-a^2} + 1 - \frac{t}{t^2-a^2}$

$v'y_h = 1 - \frac{t}{t^2-a^2} \Rightarrow \int v' = \left(1-\frac{t}{t^2-a^2}\right) y_h^{-1} = \int \left(1 - \frac{t}{t^2-a^2}\right) e^{\frac{\ln|t+a|-\ln|t-a|}{2a}} dt$

$\int u dv = uv - \int v du$

$\int \left(e^{\frac{\ln|t+a|-\ln|t-a|}{2a}} - \frac{te^{\frac{\ln|t+a|-\ln|t-a|}{2a}}}{t^2-a^2}\right) dt = \int e^{\frac{\ln|t+a|-\ln|t-a|}{2a}} dt - \int t \frac{e^{\frac{\ln|t+a|-\ln|t-a|}{2a}}}{t^2-a^2} dt$

$\int e^{\frac{\ln|t+a|-\ln|t-a|}{2a}} dt + \left(+te^{\frac{\ln|t+a|-\ln|t-a|}{2a}} - \int e^{\frac{\ln|t+a|-\ln|t-a|}{2a}} dt\right)$ $\quad u = t, \; dv = \frac{e^{\frac{\ln|t+a|-\ln|t-a|}{2a}}}{t^2-a^2} \quad du = 1, \; v = -e^{\frac{\ln|t+a|-\ln|t-a|}{2a}}$

$v = te^{\frac{\ln|t+a|-\ln|t-a|}{2a}} + c$

$y = e^{\frac{\ln|t-a|-\ln|t+a|}{2a}} \cdot \left(te^{\frac{\ln|t+a|-\ln|t-a|}{2a}} + c\right)$

$y = t + ce^{\frac{\ln|t-a|-\ln|t+a|}{2a}}$

$e^{\frac{\ln|t-a|-\ln|t+a|}{2a}} = e^{\frac{\ln\left|\frac{t-a}{t+a}\right|}{2a}} = e^{\ln\left|\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}}\right|}$

$= \left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}}$

$$\boxed{y = t + c\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}}}$$

7.1 7(a) **5 / 5**

- ✓ **- 0 pts** Correct
- **- 2 pts** y_h /IF correct, didn't find y_p
- **- 2 pts** minor mistake / gap
- **- 4 pts** Major mistake/gap
- **- 5 pts** blank
- **- 3 pts** y_h/IF minor mistake/not simplified, didn't find y_p
- **- 1 pts** y_h not simplifed
- **- 3 pts** y_h = ?
- **- 1 pts** typo

7. Let a be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2 - a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

Look at a different page provided for 7a

$$y = t + c\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}}$$

(b) (3 points) Consider the above differential equation together with the initial condition $\underline{y(a) = b}$ (initial value problem), where b is a real number. Prove that,

- if $\underline{b = a}$, there are infinite many solution to the initial value problem. (i.e. go through the initial condition.)
- if $\underline{b \neq a}$, there is no solutions to the initial value problem.

Using the solution in part (a), where $y = t + c\left(\frac{(t-a)}{(t+a)}\right)^{\frac{1}{2a}}$, $y(a) = a + c\left(\frac{a-a}{a+a}\right)^{\frac{1}{2a}} = a$. Since $y(a)$ doesn't depend on c, all solutions converge to $(a, 0)$, indicating an infinite number of solutions when $b = a$. Likewise, there are no solutions when $y(a) \neq a$, thus, no solutions when $b \neq a$. ∎

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

The above result contradicts with the existence and uniqueness theorem when $t \leq a$, because $y(t)$ cannot exist* if $t < a$ and all solutions start at $(a, 0)$, as explained in part b, contradicting the theorem at $t = a$. Otherwise, the result does not contradict the theorem when $t > a$ because y' is continuous for all $t \in (a, \infty)$.

$$y' = \frac{c\sqrt[2a]{\frac{t-a}{t+a}}}{t^2 - a^2} + 1$$

Root continuous for $t > a$; Denominator continuous for $t \neq a$ } continuous at (a, ∞).

* Cannot have a negative number inside an even root!

7.2 7(b) **3 / 3**

- ✓ **- 0 pts** **Correct:**
- **- 1 pts** minor mistake (e.g. forget to say C can be anything), gap, logic flow not clear
- **- 2 pts** some meaningful writings. not much detail provided, many gaps.
- **- 3 pts** nothing meaningful

7. Let a be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2 - a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

Look at a different page provided for 7a

$$y = t + c\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}}$$

(b) (3 points) Consider the above differential equation together with the initial condition $y(a) = b$ (initial value problem), where b is a real number. Prove that,

- if $b = a$, there are infinite many solution to the initial value problem. (i.e. go through the initial condition.)
- if $b \neq a$, there is no solutions to the initial value problem.

Using the solution in part (a), where $y = t + c\left(\frac{(t-a)}{(t+a)}\right)^{\frac{1}{2a}}$, $y(a) = a + c\left(\frac{a-a}{a+a}\right)^{\frac{1}{2a}} = a$. Since $y(a)$ doesn't depend on c, all solutions converge to $(a, 0)$, indicating an infinite number of solutions when $b = a$. Likewise, there are no solutions when $y(a) \neq a$, thus, no solutions when $b \neq a$. ∎

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

The above result contradicts with the existence and uniqueness theorem when $t \leq a$, because $y(t)$ cannot exist* if $t < a$ and all solutions start at $(a, 0)$, as explained in part b, contradicting the theorem at $t = a$. Otherwise, the result does not contradict the theorem when $t > a$ because y' is continuous for all $t \in (a, \infty)$.

$$y' = \frac{c\sqrt[2a]{\frac{t-a}{t+a}}}{t^2 - a^2} + 1$$

Root continuous for $t > a$; Denominator continuous for $t \neq a$ } continuous at (a, ∞).

*Cannot have a negative number inside an even root!

7.3 7(c) **0 / 2**

- **- 0 pts** Correct
- **- 2 pts** wrong
- ✓ **- 1 pts didn't put in normal form y' = F(y,t) = 1/ (t^2 - a^2) y + ...**
- ✓ **- 1 pts didn't check/state that \partial F/ \partial y = 1/(t^2 - a^2) or calculation is wrong**
- **- 1 pts** Gap

Last six digits of UID: 166 732

8. (10 points) Calculate e^{tA}, where $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

(Hint: Use truncation formula)

$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots$

$e^{tA}v = e^{\lambda t} e^{t(A-\lambda I)}\vec{v} = e^{\lambda t}v$

$A = aI + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$e^{tA} = e^{t(aI + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})}$

$= e^{taI} e^{\begin{pmatrix} 0 & bt \\ 0 & 0 \end{pmatrix}}$

$\det(A - \lambda I) = \lambda^2 - 2a\lambda + a^2 \rightarrow (\lambda - a)^2$

$\lambda = a$

$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(A - \lambda I)v_2 = v_1$

$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix}$

$y = ce^{at}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{at}\left(\begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$y = \begin{pmatrix} e^{at} & te^{at} \\ 0 & \frac{e^{at}}{b} \end{pmatrix}$

$y(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$

$y(0)^{-1} = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{pmatrix}$ det = 1

$e^{At} = y \cdot y(0)^{-1} = \begin{pmatrix} e^{at} & te^{at} \\ 0 & \frac{e^{at}}{b} \end{pmatrix}\begin{pmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{pmatrix}$

$$e^{tA} = \begin{pmatrix} \frac{e^{at}}{b} & te^{at} \\ 0 & \frac{e^{at}}{b} \end{pmatrix}$$

8 Question 8 **8 / 10**

- **- 0 pts** Correct
- **- 4 pts** gap: did not verify (A-aI)^2 = 0
- **- 4 pts** minor mistake
- ✓ **- 2 pts lack essential detail / some typos**
- **- 4 pts** based on your flow, you didn't use math induction to give a proof for A^n
- **- 8 pts** Major mistake