Final Exam

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Signature:

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: YIH, SAMUEL
	1C	1D	TA: KIM, BOHYUN
	1E	1F	TA: BOSCHERT, NICHOLAS

Instructions: Please print your name and student ID number above, and circle the number of your discussion section. You must **show your work** to receive credit. Please **circle or box your final answers.**

Please do not write below this line.

1. (10 points) Solve the homogeneous equation (Your final answer should be in $y = f(x, C)$ form, e.g $y = \frac{1}{C+x}$):

$$(-xy + y^2)dx + x^2dy = 0.$$

$x^2 dy = (xy - y^2) dx$

$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$

$y = vx$

$\frac{dy}{dx} = v + \frac{dv}{dx} \cdot x$

$v + x\frac{dv}{dx} = \frac{vx^2 - v^2x^2}{x^2}$

$v + x\frac{dv}{dx} = \frac{\cancel{x^2}(v - v^2)}{\cancel{x^2}}$

$x\frac{dv}{dx} = -v^2$

$\int \frac{dv}{-v^2} = \int \frac{dx}{x}$

$\frac{1}{v} = \ln x + c$

$\frac{x}{y} = \ln x + c$

$x = y(\ln x + c)$

$y = \frac{x}{\ln x + c}$

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2. (a) (5 points) Find the general solution to the differential equation:

$$y'' - 2y' + y = 0$$

Let $y = e^{\lambda t}$

$y' = \lambda e^{\lambda t}$

$y'' = \lambda^2 e^{\lambda t}$

$e^{\lambda t}(\lambda^2 - 2\lambda + 1) = 0$ $\quad e^{\lambda t} \neq 0$

$\therefore \lambda^2 - 2\lambda + 1 = 0$

$\lambda^2 - \lambda - \lambda + 1 = 0$

$\lambda(\lambda - 1) - 1(\lambda - 1) = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$ (Repeated root)

$y_h(t) = C_1 e^t + C_2 t e^t.$

(b) (10 points) Find a particular solution to the differential equation (Hint: split forcing term into two parts, check the table in P172 of your textbook):

$$y'' - 2y' + y = e^t(t+1) + e^t \sin t.$$

Equation can be split into 2 parts.

$y_1'' - 2y_1' + y_1 = e^t(t+1)$ AND $y_2'' - 2y_2' + y_2 = e^t \sin t$

Guess $y_1 = t^2(At + B)e^t$

$y_1 = (At^3 + Bt^2)e^t$

Since e^t & te^t are solutions above.

$y_1' = (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t$

$y_1'' = (6At + 2B)e^t + 2(3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t$

$(6At + 2B)e^t + \cancel{2(3At^2 + 2Bt)e^t} + \cancel{(At^3 + Bt^2)e^t} - \cancel{2(3At^2 + 2Bt)e^t}$
$- \cancel{2(At^3 + Bt^2)e^t} + \cancel{(At^3 + Bt^2)e^t} = e^t(t+1)$

$(6At + 2B)e^t = e^t(t+1)$

$6A = 1 \qquad 2B = 1$

$A = \frac{1}{6} \qquad B = \frac{1}{2}.$

$y_1 = \left(\frac{t^3}{6} + \frac{t^2}{2}\right)e^t$

Guess $y_2 = e^t(A\cos t + B\sin t)$

$y_2' = (-A\sin t + B\cos t)e^t + e^t(A\cos t + B\sin t)$

$y_2'' = (-A\cos t - B\sin t)e^t + 2(-A\sin t + B\cos t)e^t$
$+ e^t(A\cos t + B\sin t)$

$(-A\cos t - B\sin t)e^t + \cancel{2(-A\sin t + B\cos t)e^t} + \cancel{(A\cos t + B\sin t)e^t}$
$- \cancel{2(-A\sin t + B\cos t)e^t} - \cancel{2(A\cos t + B\sin t)e^t} +$
$\cancel{e^t(A\cos t + B\sin t)} = e^t \sin t$

$(-A\cos t - B\sin t)e^t = e^t \sin t$

$A = 0 \qquad B = -1$

$y_2 = e^t(-\sin t)$

$y_p(t) = \frac{t^3}{6}e^t + \frac{t^2}{2}e^t - e^t \sin t).$

3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1y_1(t) + C_2y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

trace $= 1$ $D = -2$

$A - \lambda I = \begin{pmatrix} -2-\lambda & -2 \\ 2 & 3-\lambda \end{pmatrix}$

Characteristic polynomial $= \lambda^2 - \lambda - 2$

$= \lambda^2 - 2\lambda + \lambda - 2 = 0$

$= \lambda(\lambda-2) + 1(\lambda-2) = 0$

$= (\lambda+1)(\lambda-2) = 0$

$\therefore \lambda = -1$ or $\lambda = 2$

For $\lambda = -1$

$A + I = \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix}$

$\therefore \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$y_1(t) = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda = 2$

$A - 2I = \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix}$

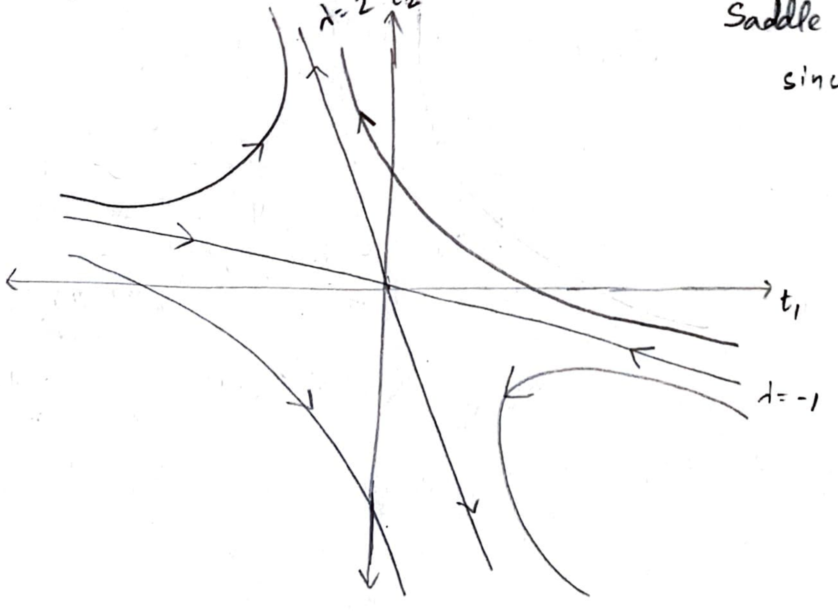
$\begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$y_2(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$y_{\text{general}} = c_1 e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)



Saddle point since $\lambda_1 < 0 < \lambda_2$

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4. (10 points) Find the solution $\mathbf{y}(t)$ to the following 3×3 system with given initial condition $\mathbf{y}(0) = (2, -2, 1)^T$:

$$\mathbf{y}' = \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}}_{A} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial)

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$p(\lambda) = (-1)^3 \det(A - \lambda I)$$

$$= -1\Big((3-\lambda)\big((1-\lambda)(2-\lambda)+1\big) - \big(-1(2-\lambda)+1\big) + \big(-1-(1-\lambda)\big)\Big)$$

$$= -1\big((3-\lambda)(1-\lambda)(2-\lambda) + (3-\lambda) - (\lambda-2) - 1 - 1 + (\lambda-1)\big)$$

$$= -1\big((3-\lambda)(1-\lambda)(2-\lambda) - \lambda + 2\big)$$

$$= -1\big((3-4\lambda+\lambda^2)(2-\lambda) - \lambda + 2\big)$$

$$= -1\big(6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 - \lambda + 2\big)$$

$$= (\lambda^3 - 6\lambda^2 + 12\lambda - 8) \qquad \frac{p}{q} \longrightarrow (\pm 1, 2, 3, 4)$$

$$= (\lambda - 2)^3 = 0$$

$\lambda = 2$, algebraic multiplicity $= 3$

$$A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ +I \\ -I \end{matrix}$$

↓ After row reduction

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Find $\vec{v_1}, \vec{v_2}, \vec{v_3}$ (linearly independent) in $\ker(A-2I)^k$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad y_1(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

This is the only linearly independent vector in $A-2I$. Therefore go to higher orders

$$(A-2I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ +I \\ \\ \end{matrix}$$

↓ after rref

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad y_2(t) = e^{2t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix} t \\ -t \\ 1 \end{pmatrix}$$

For third linearly independent vector $\vec{v_3}$ go to $(A-2I)^3$

$$(A-2I)^3 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

choose $\vec{v_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as it is linearly independent to $\vec{v_1}$ & $\vec{v_2}$

$$y_3(t) = e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix} t + \frac{t^2}{2} \\ 1 - t - \frac{t^2}{2} \\ t \end{pmatrix}$$

Continued ↗

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Question 4 - continued.

$$y(t) = C_1 e^{2t}\begin{pmatrix}1\\-1\\0\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}t\\-t\\1\end{pmatrix} + C_3 e^{2t}\begin{pmatrix}t+\frac{t^2}{2}\\1-t-\frac{t^2}{2}\\t\end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\0\\1\end{pmatrix} + C_3\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$C_1 = 2$

$-C_1 + C_3 = -2$

$C_2 = 1$

$C_1 = 2, C_2 = 1, C_3 = 0$

$$y(t) = 2e^{2t}\begin{pmatrix}1\\-1\\0\end{pmatrix} + e^{2t}\begin{pmatrix}t\\-t\\1\end{pmatrix}$$

$$y(t) = e^{2t}\begin{pmatrix}2+t\\-2-t\\1\end{pmatrix}$$

5. (15 points) Find the general solution(fundamental set) $\mathbf{y}(t)$ to the following 6×6 system:

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \mathbf{y}$$

(Hint : This is a block matrix. Try find a 1 by 1, 3 by 3, and 2 by 2 block.)

$A = (0)$ $\quad B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$

A) $y_1' = A y_1$

$A - \lambda I = (-\lambda)$

$\det(A - \lambda I) = 0$

$-\lambda = 0$

$\lambda = 0.$

$A - (0)I = (0) \quad \vec{v}_1 = (0)$

$\vec{v}_1 = (1) \quad \vec{y}_1 = (1)$

B) $y_2' = B y_2$

$B - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

$\det(B - \lambda I) = (2-\lambda)(-\lambda) + 1 = 0$

$-2\lambda + \lambda^2 + 1 = 0.$

$(\lambda - 1)^2 = 0$

$\lambda = 1$, algebraic multiplicity $= 2$.

$B - I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{y}_2 = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(B - I)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad \vec{y}_3 = e^t \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$

$\vec{y}_3 = e^t \begin{pmatrix} t \\ 1-t \end{pmatrix}$

C) $y_3' = C y_3$

$C - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -2 \\ 0 & 2 & -\lambda \end{pmatrix}$

$\det(C - \lambda I) = (-1)^3 (1-\lambda) \det\begin{pmatrix} -\lambda & -2 \\ 2 & -\lambda \end{pmatrix}$

$= (\lambda - 1)(\lambda^2 + 4) = 0$

$\lambda = 1 \quad$ or $\lambda = \pm 2i.$

$C - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 2 & -1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\vec{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{y}_4 = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$C - 2iI = \begin{pmatrix} 1-2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 2 & -2i \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\vec{v}_5 = \begin{pmatrix} 0 \\ -1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{y}_5 = e^{-2it} \left[\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

$\vec{y}_5 = (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

$\vec{y}_5 = \left(\cos 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + i \left(\sin 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \cos 2t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$\vec{y}_5 = \begin{pmatrix} 0 \\ -\cos 2t \\ -\sin 2t \end{pmatrix} \quad \vec{y}_6 = \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix}$

Fundamental set

$$y(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^t \\ -e^t \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^t(t) \\ e^t(1-t) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^t \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\cos 2t \\ -\sin 2t \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix}$$

6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

$y' = \sin y + \cos y$ (autonomous equation)

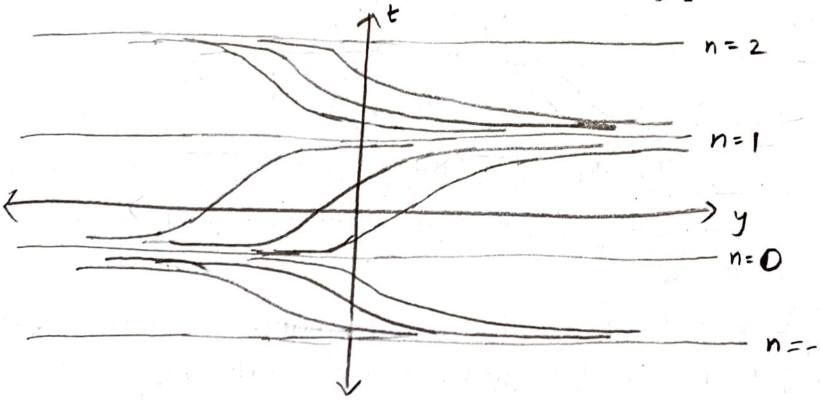
$\sin y + \cos y = 0$

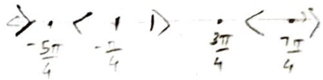
$\sin y = -\cos y$

$\tan y = -1$

$y = \left(\frac{4n-1}{4}\right)\pi$ where $n \in \mathbb{Z}$

(b) (5 points) Sketch the solutions on the $t - y$ plane.





(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

Consider $f(y) = \sin y + \cos y$

$f(y)$ is continuous on $(-\infty, \infty)$

$\frac{\partial f}{\partial y} = \cos y - \sin y$ which is continuous on $(-\infty, \infty)$

Consider a rectangle R which is $\mathbb{R}^2$

Both $f(y)$ & $\frac{\partial f}{\partial y}$ continuous on R therefore a unique solution exists which goes through all of R

Consider two trivial solutions $y_1(t) = m$ & $y_2(t) = M$ where $m < M$ & $m, M \in \mathbb{R}$

By IVT, $\exists$ a $t \in (-\infty, \infty)$ such that $m < y(t) < M$

Since the solutions $y(t)$, $y_1(t)$ & $y_2(t)$ never cross each other in R according to the uniqueness theorem, $m < y(t) < M$ for all $t \in (-\infty, \infty)$

7. Let a be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2 - a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

$(t^2-a^2)y' - (t^2-a^2) = y - t$

$(t^2-a^2)(y'-1) = y-t$

Let $f = y - t$

$\frac{df}{dt} = y' - 1$

$(t^2-a^2)\frac{df}{dt} = f$

$\frac{df}{dt} - \frac{f}{t^2-a^2} = 0$

Integrating factor $= e^{-\int\frac{1}{(t+a)(t-a)}}$

Partial Fraction $1 = A(t-a) + B(t+a)$

$B = \frac{1}{2a}$ $A = -\frac{1}{2a}$

$e^{\int\left(\frac{1}{2a(t+a)} - \frac{1}{2a(t-a)}\right)dt}$

$e^{\frac{1}{2a}\int\left(\frac{1}{t+a} - \frac{1}{t-a}\right)dt}$

$e^{\frac{1}{2a}\ln\left(\frac{t+a}{t-a}\right)}$

$IF = \left(\frac{t+a}{t-a}\right)^{\frac{1}{2a}}$

$\left(\frac{t+a}{t-a}\right)^{\frac{1}{2a}}\left(\frac{df}{dt} - \frac{f}{t^2-a^2}\right) = 0$

$\left(\left(\frac{t+a}{t-a}\right)^{1/2a} f\right)' = 0$

$\left(\frac{t+a}{t-a}\right)^{1/2a} f = \int 0\, dt$

$\left(\frac{t+a}{t-a}\right)^{1/2a} f = 0 + C$

$f = C\left(\frac{t-a}{t+a}\right)^{1/2a}$

$y - t = C\left(\frac{t-a}{t+a}\right)^{1/2a}$

$y(t) = t + C\left(\frac{t-a}{t+a}\right)^{1/2a}$

$a \neq 0$

(b) (3 points) Consider the above differential equation together with the initial condition $y(a) = b$ (initial value problem), where b is a real number. Prove that,

- if $b = a$, there are infinite many solution to the initial value problem. (i.e. go through the initial condition.)
- if $b \neq a$, there is no solutions to the initial value problem.

For initial condition $f(a) = b$

$b = a + C\left(\frac{a-a}{a+a}\right)^{\frac{1}{2a}}$

$b = a + 0$

$b = a$

If $a \neq b$, then no solutions

If $a = b$ then $y(t) = t + C\left(\frac{t-a}{t+a}\right)^{1/2a}$ holds true for $\forall c \in \mathbb{R}$

and infinite solutions

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

$y' = \frac{y + t^2 - t - a^2}{t^2 - a^2} = f(t,y)$

Clearly $f(t,y)$ is not continuos at $t = a$

Thus, $f(t,y)$ is not continuous on any rectangle R in t-y plane that contains $(t,y) = (a,y)$

Therefore the existence theorem doesn't apply here as $f(t,y)$ is discontinuous on any R containing the intial condition. Since existence theorem doesn't apply, uniqveness also doesn't apply and the weird result doesn't contradict the two theorems.

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8. (10 points) Calculate e^{tA}, where $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

(Hint: Use truncation formula)

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B}$$

$$A = aI + bB$$

$$e^{tA} = e^{t(aI + Bb)}$$

$$= e^{aIt} \cdot e^{Bbt}$$

$$= e^{at}\left[I + btB + \frac{b^2t^2}{2!}B^2 + \frac{b^3t^3B^3}{3!} + \dots\right]$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B^3, B^4, \dots = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Can be simplified to

$$e^{tA} = e^{at}\left[I + btB\right]$$

$$= e^{at}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & bt \\ 0 & 0 \end{pmatrix}\right]$$

$$= e^{at}\left[\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}\right]$$

$$e^{tA} = e^{at}\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}$$