Final Exam

Last Name: ______________________________

First Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Section:

Tuesday:	Thursday:	
1A	1B	TA: YIH, SAMUEL
1C	1D	TA: KIM, BOHYUN
1E	1F	TA: BOSCHERT, NICHOLAS

Instructions: Please print your name and student ID number above, and circle the number of your discussion section. You must **show your work** to receive credit. Please **circle or box your final answers.**

Please do not write below this line.

1. (10 points) Solve the homogeneous equation (Your final answer should be in $y = f(x, C)$ form, e.g $y = \frac{1}{C+x}$):

$$(-xy + y^2)dx + x^2 dy = 0.$$

2. (a) (5 points) Find the general solution to the differential equation:

$$y'' - 2y' + y = 0$$

(b) (10 points) Find a particular solution to the differential equation (Hint: split forcing term into two parts, check the table in P172 of your textbook):

$$y'' - 2y' + y = e^t(t+1) + e^t \sin t.$$

3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)

4. (10 points) Find the solution $\mathbf{y}(t)$ to the following 3×3 system with given initial condition $\mathbf{y}(0) = (2, -2, 1)^T$:

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial)

5. (15 points) Find the general solution(fundamental set) $\mathbf{y}(t)$ to the following 6×6 system:

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \mathbf{y}$$

(Hint : This is a block matrix. Try find a 1 by 1, 3 by 3, and 2 by 2 block.)

6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

(b) (5 points) Sketch the solutions on the $t - y$ plane.

(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

7. Let a be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2 - a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

(b) (3 points) Consider the above differential equation together with the initial condition $y(a) = b$ (initial value problem), where b is a real number. Prove that,

- if $b = a$, there are infinite many solution to the initial value problem. (i.e. go through the initial condition.)
- if $b \neq a$, there is no solutions to the initial value problem.

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

8. (10 points) Calculate e^{tA}, where $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

(Hint: Use truncation formula)