Zicong

1. (25 points) Solve the equation system below with initial conditions by steps (a) - (c).

$$\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = x(t) \end{cases} \quad \text{with} \quad \begin{cases} x(0) = 1 \\ y(0) = 2 \end{cases}$$

(a) (5 points) Show the second order equation that is satisfied by $y(t)$;

(b) (5 points) Show the corresponding initial conditions of this equation;

(c) (10 points) Solve the function $y(t)$ from the previous step;

(d) (5 points) Solve the function $x(t)$.

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$x(t) = y'(t)$

$x'(t) = y''(t) = 2y'(t) - y(t)$

$x(0) = y'(0) = 1$

$y(0) = 2$

$y'' = 2y' - y \rightarrow y'' - 2y' + y = 0$

Characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda = -1$

$y_1 = e^{-t}, \quad y_2 = te^{-t}$

$y(t) = C_1 e^{-t} + C_2 te^{-t}$

$y(0) = C_1 + C_2 \cdot 0 = C_1 = 2$

$y'(0) = -C_1 e^{-0} + C_2 (e^{-t} - te^{-t}) = -C_1 + C_2 = 1 \rightarrow C_2 = 3$

$y(t) = 2e^{-t} + 3te^{-t}$

$x(t) = y'(t)$

$= \frac{d}{dt}(2e^{-t} + 3te^{-t})$

$= -2e^{-t} + 3(e^{-t} - te^{-t}) = e^{-t} - 3te^{-t}$

$T = \frac{2\pi}{\omega} \to \omega$

2. (25 points) A 0.1-kg mass is attached to a spring having a spring constant 3.6 kg/s^2. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present. Let $x(t)$ be the displacement of the mass at t.

   (a) (5 points) Show the differential equation satisfied by $x(t)$ and its initial conditions.

   (b) (5 points) Solve the function $x(t)$.

   (c) (5 points) What is the **amplitude** of the motion?

   (d) (5 points) What is the **frequency** of the motion?

   (e) (5 points) What is the **phase** of the motion?

$m = 0.1$ kg

$k = 3.6$ kg/s^2

$\mu = 0$

$f(t) = 0$

a) $mx''(t) + \mu x'(t) + kx(t) = f(t)$

5 $0.1x'' + 3.6x = 0$

$x(0) = 0$

$x'(0) = -0.4$

b) $0.1x'' + 3.6x = 0 \to x'' + 36x = 0$

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5 Characteristic equation: $\lambda^2 + 36 = 0$, $\lambda = \pm 6i$

$x_1 = e^0 \cos 6t = \cos 6t$

$x_2 = e^0 \sin 6t = \sin 6t$

$x(t) = C_1 \cos 6t + C_2 \sin 6t$

$x(0) = C_1 \cos(6 \cdot 0) + C_2 \sin(6 \cdot 0) = C_1 = 0$

$x'(0) = -6C_1 \sin(6 \cdot 0) + 6C_2 \cos(6 \cdot 0) = 6C_2 = -0.4 \to C_2 = \frac{-0.4}{6} = \frac{-4}{60} = \frac{-1}{15}$

$\boxed{x(t) = \frac{-1}{15} \sin 6t}$

c) $A = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{1}{15}\right)^2} = \boxed{\frac{1}{15} \text{ m}}$

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d) 5 Angular frequency: $\omega = 6$ rad/s $\to \omega = 2\pi f \to$ Frequency $= \boxed{\frac{3}{\pi} \text{ Hz}}$

e) Phase:

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3. (25 points) Solve the following differential equation by steps.

$$y'' + 4y' + 4y = 5e^{-2t} + 2\sin(2t) + 3t + 4 \tag{1}$$

(a) (5 points) Find the general solution to the associated homogeneous equation.

(b) (5 points) Find a particular solution $y_{p_1}(t)$ to the equation: $y'' + 4y' + 4y = e^{-2t}$.

(c) (5 points) Find a particular solution y_{p_2} to the equation: $y'' + 4y' + 4y = \sin(2t)$.

(d) (5 points) Find a particular solution y_{p_3} to the equation: $y'' + 4y' + 4y = 3t + 4$.

(e) (5 points) Find the expression of the general solution $y(t)$ to the equation (1).

e homogeneous equation has characteristic equation $\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0 \rightarrow \lambda = -2$ $\quad y_1 = e^{-2t}, \; y_2 = te^{-2t}$

$\boxed{y = c_1 e^{-2t} + c_2 te^{-2t}}$ ✓

Let $y_{p_1}(t) = ae^{-2t}$

$y''(t) + 4y_{p_1}'(t) + 4y_{p_1}(t) = 4ae^{-2t} - 8ae^{-2t} + 4ae^{-2t} = 0$

nce ae^{-2t} and te^{-2t} are solutions to the homogeneous equation, let $y_{p_1}(t) = at^2e$

$y_{p_1} = at^2e^{-2t}$ ✓

$y_{p_1}' = a(2te^{-2t} - 2t^2e^{-2t}) = 2ae^{-2t}(t - t^2)$

$y_{p_1}'' = 2a(-2e^{-2t}(t-t^2) + e^{-2t}(1-2t)) = 2a(-2te^{-2t} + 2t^2e^{-2t} + e^{-2t} - 2te^{-2t}) = 2ae^{-2t}(2t^2$

$y_{p_1}'' + 4y_{p_1}' + 4y_{p_1} = ae^{-2t}(4t^2 - 8t + 2) + ae^{-2t}(8t - 8t^2) + ae^{-2t} \cdot 4t^2 = e^{-2t}$ $\quad$ Since $e^{-2t} \neq 0$

$4at^2 - 8at + 2a + 8at - 8at^2 + 4at^2 = 1$ ✓

$2a = 1 \rightarrow a = 1/2$

$\boxed{y_{p_1} = \frac{1}{2}t^2e^{-2t}}$ ✓

$y_{p_2} = a\cos 2t + b\sin 2t$

$y_{p_2}' = -2a\sin 2t + 2b\cos 2t$

$y_{p_2}'' = -4a\cos 2t - 4b\sin 2t$

$y_{p_2}'' + 4y_{p_2}' + 4y_{p_2} = (-4a + 8b + 4a)\cos 2t + (-4b - 8a + 4b)\sin 2t = \sin 2t$

✓ $\quad 8b = 0 \quad -8a = 1$

$\boxed{y_{p_2}(t) = \frac{-1}{8}\cos(2t)}$

$y_{p_3} = at + b \rightarrow 0 + 4a + 4(at + b) = 3t + 4 \rightarrow a = 3/4, \; b = 1/4$

4. (25 points) Solve the general solution of the following equation by steps.

$$y'' + y' + y = 2t\sin(t) \qquad (2)$$

(a) (10 points) Find constants A and B, such that the function $z(t) = (At + B)e^{it}$ solves the equation: $z'' + z' + z = te^{it}$.

(b) (10 points) Find a particular solution $y_p(t)$ to the equation (2).

(c) (5 points) Find the general solution to the equation (2).

$z(t) = (At+B)e^{it}$

$z'(t) = (Ae^{it}) + ie^{it}(At+B) = e^{it}(iAt+iB+A)$

$z''(t) = ie^{it}(iAt+iB+A) + e^{it}(iA) = e^{it}(-At-B+Ai+iA) = e^{it}(-At-B+2iA)$

$z''(t)+z'(t)+z(t) = e^{it}(-At-B+2iA+iAt+iB+A+At+B) = te^{it}$ Since $e^{it} \neq$

$(2iA + iAt + iB + A) = t$

$iAt = t \rightarrow A = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$

$2iA + iB + A = 0 \rightarrow 2i(-i) + iB - i = 0 \rightarrow iB = i - 2 \rightarrow B = 1 - \frac{2}{i} =$

$\boxed{z(t) = (-it + (1+2i))e^{it}}$

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$y_p = 2\operatorname{Im}(z(t))$:

$z(t) = (-it + (1+2i))e^{it} = (-it + (1+2i))(\cos t + i\sin t) = -it\cos t + t\sin t + (1+2i)\cos t + (1-$

$z(t) = (t\sin t + \cos t - 2\sin t) + i(-t\cos t + 2\cos t + \sin t)$

$y_p = \operatorname{Im}(z(t)) = -t\cos t + 2\cos t + \sin t$

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Homogeneous equation: $y'' + y' + y = 0$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$y_1 = e^{-1/2t}\cos(\sqrt{3}/2\, t)$

$y_2 = e^{-1/2t}\sin(\sqrt{3}/2\, t)$

$y_h = C_1 y_1 + C_2 y_2$