1. (25 points) Solve the equation system below with initial conditions by steps (a) - (c).

25

$$\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = x(t) \end{cases} \quad \text{with} \quad \begin{cases} x(0) = 1 \\ y(0) = 2 \end{cases}$$

(a) (5 points) Show the second order equation that is satisfied by $y(t)$;

(b) (5 points) Show the corresponding initial conditions of this equation;

(c) (10 points) Solve the function $y(t)$ from the previous step;

(d) (5 points) Solve the function $x(t)$.

a) $x'(t) = y''(t)$

$y''(t) = 2y'(t) - y(t)$

$y''(t) - 2y'(t) + y(t) = 0$

b) $x(0) = 1 = y'(0)$

$y(0) = 2$

$\cancel{y''(0) - 2(1) + 2 = 0}$

$\cancel{y''(0) = 0}$

c) $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)(\lambda - 1) = 0$

$\lambda = 1$

$y(t) = C_1 e^t + C_2 t e^t$

$y'(t) = C_1 e^t + C_2 e^t + C_2 t e^t$

$y(0) = 2 = C_1$

$y'(0) = 1 = C_1 + C_2$

$C_2 = 1 - 2 = -1$

$\boxed{y(t) = 2e^t - te^t}$

d) $x(t) = y'(t)$

$x(t) = 2e^t - e^t - te^t$

$\boxed{x(t) = e^t - te^t}$

2. (25 points) A 0.1-kg mass is attached to a spring having a spring constant 3.6 kg/s^2. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present. Let $x(t)$ be the displacement of the mass at t.

   (a) (5 points) Show the differential equation satisfied by $x(t)$ and its initial conditions.

   (b) (5 points) Solve the function $x(t)$.

   (c) (5 points) What is the **amplitude** of the motion?

   (d) (5 points) What is the **frequency** of the motion?

   (e) (5 points) What is the **phase** of the motion?

a) $0.1x''(t) + 3.6x(t) = 0$

5

$x(0) = 0$

$x'(0) = -0.4$

b) $0.1\lambda^2 + 3.6 = 0$

5

$\lambda^2 + 36 = 0$

$\lambda = \frac{\pm\sqrt{-4(36)}}{2}$

20

$\lambda = \pm 6i$

$x(t) = C_1 \cos 6t + C_2 \sin 6t$

$x(0) = 0 = C_1$

$x'(t) = -6C_1 \sin 6t + 6C_2 \cos 6t$

$x'(0) = -0.4 = 6C_2$

$C_2 = \frac{-1}{15}$

$\boxed{x(t) = \frac{-1}{15}\sin 6t}$

5 c) Amplitude $= \frac{1}{15}$

5 d) ω_0 (frequency) $= 6$

e) ϕ (phase) $= 0$

Flip Flops

3. (25 points) Solve the following differential equation by steps.

20

$$y'' + 4y' + 4y = 5e^{-2t} + 2\sin(2t) + 3t + 4 \quad (1)$$

(a) (5 points) Find the general solution to the associated homogeneous equation.

(b) (5 points) Find a particular solution $y_{p_1}(t)$ to the equation: $y'' + 4y' + 4y = e^{-2t}$.

(c) (5 points) Find a particular solution y_{p_2} to the equation: $y'' + 4y' + 4y = \sin(2t)$.

(d) (5 points) Find a particular solution y_{p_3} to the equation: $y'' + 4y' + 4y = 3t + 4$.

(e) (5 points) Find the expression of the general solution $y(t)$ to the equation (1).

5

a) $y_h'' + 4y_h' + 4y_h = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0$

$\lambda = -2$

$\boxed{y_h(t) = e^{-2t}(C_1 + tC_2)}$

b) $y'' + 4y' + 4y = e^{-2t}$

$y_{p_1}(t) = Ae^{-2t}$

$4 + (-8) + 4 = 0$

$\Rightarrow y_{p_1}(t) = Ate^{-2t}$

$y_{p_1}'(t) = Ae^{-2t} - 2Ate^{-2t}$

$y_{p_1}''(t) = -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t}$

$\Rightarrow e^{-2t}(-2A - 2A + 4At + 4A - 8At + 4At) = e^{-2t}$

$\Rightarrow 0 = 0$

$\Rightarrow y_{p_1}(t) = At^2e^{-2t}$

$y_{p_1}'(t) = 2Ate^{-2t} - 2At^2e^{-2t}$

$y_{p_1}''(t) = 2Ae^{-2t} - 4Ate^{-2t} - 4Ate^{-2t} + 4At^2e^{-2t}$

$\Rightarrow e^{-2t}(2A - 4At - \cancel{4At} + 4At^2 + 8At - 8At^2 + 4At^2) = e^{-2t}$

$2A = 1$

$A = 1/2$

$\boxed{y_{p_1}(t) = \frac{1}{2}t^2e^{-2t}}$

5

c) $y'' + 4y' + 4y = \sin(2t)$

$z'' + 4z' + 4z = e^{2it}$

$z_p(t) = Ae^{2it}$

$z_p'(t) = 2iAe^{2it}$

$z_p''(t) = -4Ae^{2it}$

$e^{2it}(-4A + 8iA + 4A) = e^{2it}$

$8iA = 1$

$A = \frac{-1}{8}i$

$z_p(t) = \frac{-1}{8}ie^{2it}$

$z_p(t) = \frac{-1}{8}i(\cos 2t + i\sin 2t)$

$z_p(t) = \frac{1}{8}\sin 2t - \frac{1}{8}i\cos 2t$

$\boxed{y_{p_2}(t) = \frac{-1}{8}\cos 2t}$

5

d) $y'' + 4y' + 4y = 3t + 4$

$y_{p_3}(t) = at + b$

$y_{p_3}'(t) = a$

$y_{p_3}''(t) = 0$

$4a + 4at + 4b = 3t + 4$

$4a = 3, \quad a = 3/4$

$4a + 4b = 4$

$b = 1 - a = 1/4$

$\boxed{y_{p_3}(t) = \frac{3}{4}t + \frac{1}{4}}$

e) $y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$

$\boxed{y(t) = e^{-2t}(C_1 + tC_2) + \frac{1}{2}t^2e^{-2t} - \frac{1}{8}\cos 2t + \frac{3}{4}t + \frac{1}{4}}$

4. (25 points) Solve the general solution of the following equation by steps.

$$y'' + y' + y = 2t\sin(t) \tag{2}$$

(a) (10 points) Find constants A and B, such that the function $z(t) = (At+B)e^{it}$ solves the equation: $z'' + z' + z = te^{it}$.

(b) (10 points) Find a particular solution $y_p(t)$ to the equation (2).

(c) (5 points) Find the general solution to the equation (2).

a) $z'' + z' + z = te^{it}$

$z(t) = (At+B)e^{it}$ $z' = (At+B)ie^{it}$

$z'(t) = Ae^{it} + (At+B)e^{it}$

$z''(t) = Ae^{it} + Ae^{it} + (At+B)e^{it}$ 4/10

$\Rightarrow e^{it}(A + A + At + B + A + At + B + At + B) = te^{it}$

$3At + 3A + 3B = t$

$3A = 1$

$A = 1/3$

$3A + 3B = 0$

$B = -A = -1/3$

$\boxed{z(t) = (\frac{1}{3}t - \frac{1}{3})e^{it}}$

b) $z(t) = (\frac{1}{3}t - \frac{1}{3})(\cos t + i\sin t)$

$z(t) = (\frac{1}{3}t\cos t - \frac{1}{3}\cos t) + i(\frac{1}{3}t\sin t - \frac{1}{3}\sin t)$

$y_p(t) = 2 \cdot$ imaginary part of $z(t)$ +4. 10

$\boxed{y_p(t) = \frac{2}{3}t\sin t - \frac{2}{3}\sin t}$ not right

c) $y_h'' + y_h' + y_h = 0$

$\lambda^2 + \lambda + 1 = 0$ +T.

$\lambda = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2}$

$\lambda = \frac{-1 \pm 3i}{2}$

$\lambda = \frac{-1}{2} \pm \frac{3}{2}i$

$y_h(t) = e^{-\frac{1}{2}t}(C_1\cos\frac{3}{2}t + C_2\sin\frac{3}{2}t)$

General Solution:

$y(t) = y_h(t) + y_p(t)$

$$\boxed{y(t) = e^{-\frac{1}{2}t}(C_1\cos\frac{3}{2}t + C_2\sin\frac{3}{2}t) + \frac{2}{3}t\sin t - \frac{2}{3}\sin t}$$

18/25