# Fall 2017: Math 33B Midterm - II

**This is a closed book test. Do all work on the sheets provided.**

Grade Table (for teacher use only)

| Question | Points | Score |
|---|---|---|
| 1 | 25 | 25 |
| 2 | 25 | 24 |
| 3 | 25 | 25 |
| 4 | 25 | 25 |
| Total: | 100 | 100 |

1. (25 points) Solve the equation system below with initial conditions by steps (a) - (c).

25

$$\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = x(t) \end{cases} \quad \text{with} \quad \begin{cases} x(0) = 1 \\ y(0) = 2 \end{cases}$$

(a) (5 points) Show the second order equation that is satisfied by $y(t)$;

(b) (5 points) Show the corresponding initial conditions of this equation;

(c) (10 points) Solve the function $y(t)$ from the previous step;

(d) (5 points) Solve the function $x(t)$.

a) $x = y'$

$x' = 2x - y \rightarrow y'' = 2y' - y$

$\boxed{y'' - 2y' + y = 0}$

b) $x(0) = 1$

$\boxed{y'(0) = 1, \ y(0) = 2}$

c) $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)(\lambda - 1) = 0$

$\lambda_1 = \lambda_2 = 1$ repeated root

$y(t) = C_1 e^t + C_2 t e^t$

$y'(t) = C_1 e^t + C_2 e^t + C_2 t e^t$

$y(0) = 2$, $y'(0) = 1$ $\rightarrow$ $2 = C_1$, $1 = C_1 + C_2$

$C_1 = 2, \ C_2 = -1$

$\boxed{y(t) = 2e^t - te^t}$

d) $x(t) = y'(t)$

$= 2e^t - e^t - te^t$

$x(t) = \boxed{e^t - te^t}$

2. (25 points) A 0.1-kg mass is attached to a spring having a spring constant 3.6 kg/s². The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present. Let $x(t)$ be the displacement of the mass at $t$.

   (a) (5 points) Show the differential equation satisfied by $x(t)$ and its initial conditions.

   (b) (5 points) Solve the function $x(t)$.

   (c) (5 points) What is the **amplitude** of the motion?

   (d) (5 points) What is the **frequency** of the motion?

   (e) (5 points) What is the **phase** of the motion?

5 a) $mx'' + \mu x' + kx = 0$

$m = 0.1 \quad , \quad k = 3.6 \quad , \quad \mu = 0$

$0.1x''(t) + 3.6x(t) = 0$

$\boxed{\begin{array}{l} x''(t) + 36x(t) = 0 \\ \text{Initial conditions: } x(0) = 0 \ , \ x'(0) = -0.4 \end{array}}$ ✓

5 b) $x'' + 36x = 0$

$\lambda^2 + 36 = 0$

$\lambda = \dfrac{\pm\sqrt{-4(36)}}{2} = \pm\dfrac{12i}{2} = \pm 6i$

$x(t) = e^{0t}(C_1 \cos 6t + C_2 \sin 6t)$

$x'(t) = -6C_1 \sin 6t + 6C_2 \cos 6t$

$x(0) = 0 \rightarrow 0 = C_1$

$x'(0) = -0.4 \rightarrow -0.4 = 6C_2$

$C_1 = 0 \qquad C_2 = -\dfrac{0.4}{6} = -\dfrac{1}{15}$

✓ $\boxed{x(t) = -\frac{1}{15}\sin 6t}$

5 c) $x(t) = C_1 \cos 6t + C_2 \sin 6t$

$A = \sqrt{C_1^2 + C_2^2} = \sqrt{(-\frac{1}{15})^2} = \boxed{\frac{1}{15}\text{ m}}$

$\omega_0 = 2\pi f$

angular freq

4 d) $\omega_0 = \sqrt{\frac{k}{m}}$ ✓ $\qquad f = \dfrac{\omega_0}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{3.6}{0.1}} = \boxed{\dfrac{3}{\pi}}\ s^{-1}$

5 e) $x(t) = -\frac{1}{15}\cos(6t - \frac{\pi}{2})$ $\qquad \boxed{\phi = \frac{\pi}{2}}$

(24)

25

3. 5 (25 points) Solve the following differential equation by steps.

$$y'' + 4y' + 4y = 5e^{-2t} + 2\sin(2t) + 3t + 4 \qquad (1)$$

(a) (5 points) Find the general solution to the associated homogeneous equation.

(b) (5 points) Find a particular solution $y_{p_1}(t)$ to the equation: $y'' + 4y' + 4y = e^{-2t}$.

(c) (5 points) Find a particular solution $y_{p_2}$ to the equation: $y'' + 4y' + 4y = \sin(2t)$.

(d) (5 points) Find a particular solution $y_{p_3}$ to the equation: $y'' + 4y' + 4y = 3t + 4$.

(e) (5 points) Find the expression of the general solution $y(t)$ to the equation (1).

5

a) homogeneous:

$y_H'' + 4y_H' + 4y_H = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0$

$\lambda_1 = \lambda_2 = -2 \rightarrow \boxed{y = c_1 e^{-2t} + c_2 t e^{-2t}}$

5

b) $y'' + 4y' + 4y = e^{-2t}$

let $y_{p_1}(t) = At^2 e^{-2t}$, since $Ate^{-2t}$ and $Ae^{-2t}$ are solutions to the homogeneous equation

$y_p'(t) = 2Ate^{-2t} - 2At^2e^{-2t}$

$y_1''(t) = 2Ae^{-2t} - 4Ate^{-2t} - 2(2Ate^{-2t} - 2At^2e^{-2t})$

$= 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t}$

$\rightarrow 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t} + 4(2Ate^{-2t} - 2At^2e^{-2t}) + 4At^2e^{-2t} =$

$2Ae^{-2t} = e^{-2t}$

$2A = 1, \ A = \frac{1}{2}$

$\boxed{y_{p_1}(t) = \frac{1}{2}t^2e^{-2t}}$ 4

5

c) let $y_{p_2}(t) = a\cos 2t + b\sin 2t$

$y_{p_2}'(t) = -2a\sin 2t + 2b\cos 2t \qquad y_{p_2}''(t) = -4a\cos 2t - 4b\sin 2t$

$y'' + 4y' + 4y = \sin 2t$

$\rightarrow (-4a\cos 2t - 4b\sin 2t) + 4(-2a\sin 2t + 2b\cos 2t) + 4(a\cos 2t + b\sin 2t) = 0$

$8b\cos 2t - 8a\sin 2t = \sin 2t$

$-8a = 1 \qquad 8b = 0$

$a = -\frac{1}{8} \qquad b = 0$

$\boxed{y_{p_2}(t) = -\frac{1}{8}\cos 2t}$

more answers on next page

d) $y'' + 4y' + 4y = 3t + 4$

let $y_{p_3}(t) = At + B$

$y_{p_3}'(t) = A \quad y_{p_3}''(t) = 0$

$4A + 4(At + B) = 3t + 4$

$4A + 4B + 4At = 3t + 4$

$4A = 3, \quad 4A + 4B = 4$

$A = \frac{3}{4} \qquad B = \frac{1}{4}$

$y_{p_3}(t) = \frac{3}{4}t + \frac{1}{4}$

$y(t) = y_H(t) + 5y_{p_1}(t) + 2y_{p_2}(t) + y_{p_3}(t)$

$$\boxed{y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{5}{2}t^2 e^{-2t} + \frac{1}{4}\cos 2t + \frac{3}{4}t + \frac{1}{4}}$$

4. **(25 points) Solve the general solution of the following equation by steps.**

$$y'' + y' + y = 2t\sin(t) \qquad (2)$$

(a) (10 points) Find constants $A$ and $B$, such that the function $z(t) = (At+B)e^{it}$ solves the equation: $z'' + z' + z = te^{it}$.

(b) (10 points) Find a particular solution $y_p(t)$ to the equation (2).

(c) (5 points) Find the general solution to the equation (2).

$z'' + z' + z = te^{it}$

a) $z(t) = (At+B)e^{it}$ $\quad z'(t) = Ae^{it} + (At+B)ie^{it}$

$z''(t) = Aie^{it} + Aie^{it} - (At+B)e^{it}$

$2Aie^{it} - (At+B)e^{it} + Ae^{it} + (At+B)ie^{it} + (At+B)e^{it} = te^{it}$

$e^{it}(2Ai - (At+B) + A + (At+B)i + (At+B)) = te^{it}$

$2Ai + Bi + A + Ait = t$

$Ai = 1 \qquad 2Ai + Bi + A = 0$

$\boxed{A = -i} \qquad B = \frac{-2Ai - A}{i} = \frac{-2 - \frac{1}{i}}{i}$

10

$= -\frac{2}{i} + 1$

$\boxed{B = 2i + 1}$

b) $z(t) = (-it + 2i + 1)e^{it}$

$= (-it + 2i + 1)(\cos t + i\sin t)$

$= (-t\cos t + 2\cos t + \sin t)i + Re(z(t))$

$y_p(t) = 2\,Im(z(t))$

$y_p(t) = \boxed{-2t\cos t + 4\cos t + 2\sin t}$

10

c) $y(t) = y_H(t) + y_p(t)$

characteristic equation $y'' + y' + y = 0$ $\rightarrow$ $\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$y_H(t) = e^{-\frac{1}{2}t}(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t)$

5

$\boxed{y(t) = e^{-\frac{1}{2}t}(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t) - 2t\cos t + 4\cos t + 2\sin t}$