25

1. (25 points) The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 600 mg of Iodine 131, how much of the isotope will be left after 30 days?

Let $x(t)$ be the number of grams of $I_{131}$ after $t$ days.

The value of $x(t)$ is determined by the differential equation

$x'(t) = -\lambda x(t)$, where $\lambda$ is some constant.

This has solution: $\frac{x'(t)}{x(t)} = -\lambda$

$$\ln|x(t)| = -\lambda t + c$$

$$x(t) = x_0 e^{-\lambda t}$$

Since the initial weight is 600 mg, $x(0) = 600$ mg

$$x(0) = x_0 e^{-\lambda \cdot 0} = 600 \text{ mg}$$

$$x_0 = 600 \text{ mg}$$

To solve for $\lambda$, note that $t_{1/2}$ is the time it takes for $x$ to equal $\frac{1}{2}x_0$:

$$x(t_{1/2}) = 600 e^{-\lambda \cdot 8.04} = 300$$

$$(-8.04\lambda) = -\ln 2$$

$$\lambda = \frac{\ln 2}{8.04}.$$

Finally, the amount of isotope left after 30 days is:

$$x(30) = 600 e^{\frac{-\ln 2}{8.04} \cdot 30}$$

$$= \boxed{45.178 \text{ mg}}$$

5

2. (25 points) A 100-gal tank initially contains 40 gal of pure water. Sugar-water solution containing 2 lb of sugar for each gallon of water begins entering the tank at a rate of 4 gal/min. **After 10 minutes,** a drain is opened at the bottom of the tank, allowing the sugar-water solution to leave the tank at a rate of 2 gal/min. What is the sugar content (lb) in the tank at the precise moment that the tank is full of sugar-water solution?

Let $A(t)$ be the amount of sugar for the first 10 min

$A'(t) = (2\ \text{lb/gallon} \cdot 4\ \text{gal/min}) = 8\ \text{gal/min}.$

Let $x(t)$ be the amount of sugar in the tank, where $t$ is number of minutes since 10 minutes.

At $t=0$, the tank already has $40 + 4 \cdot 10 = 80$ gal of liquid, continues filling at a rate of $(4-2) = 2$ gal/minute. The volume in the tank is then $V(t) = 80 + 2t$. The tank is full of sugar-water solution at $t = 10$.

$x(t)$ is described by the following differential equation:

$$x'(t) = 8\ \text{gal/min} - \frac{2}{80+2t} x(t)$$

$$x'(t) + \frac{2}{80+2t} x(t) = 8 \qquad u' = au \rightarrow u' = \frac{2}{80+2t} u =$$

$$u = e^{\ln(40+t)} =$$

$$((40+t)(x(t)))' = 8(40+t)$$

$$(40+t)\,x(t) = \int 320 + 8t\, dt = 4t^2 + 320t + C$$

$$x(t) = \frac{4t^2 + 320t + C}{40+t} \qquad x(0) = \frac{2\text{lb}}{\text{gallon}} \cdot \frac{4\text{gal}}{\text{min}} \cdot 10\text{ min} = 8$$

$$x(0) = \frac{C}{40} = 80 \rightarrow C = 3200$$

Finally, the sugar content at $t=10$ is:

$(1-\frac{x}{y})dx + (\frac{x}{y^2} - \frac{1}{y^2})$

 $\frac{x}{y^2}$ $\frac{1}{y}$

3. (25 points) Solve the following differential equation:

$$(y^2 - xy)dx + (xy - 1)dy = 0$$

We first check if the equation is exact:

$\frac{\partial}{\partial y}(y^2 - xy) = 2y - x \neq \frac{\partial}{\partial x}(xy-1) = y$

Then there must be some integrating factor $\mu$ such

Suppose that $\mu$ is a function of only $x$ $(\mu = \mu(x$

Then! $\frac{\partial}{\partial y}\mu P = (y^2 - xy) + (2y - x)\mu(x)$

$\frac{\partial}{\partial x}\mu Q = \mu'(x)(xy-1) + (y)\mu(y)$

$y^2 - xy + (2y - x)\mu(x) = \mu'(x)(xy-1) + y\mu(y)$

$y(y-x) + (y-x)\mu(x) = \mu'(x)(xy-1)$

$(y-x)(y+\mu(x)) = \mu'(x)(xy-1)$

Since in this equation $\mu'(x)$ is also a function o

not be a function of only $x$.

Suppose $\mu$ is a function of only $y$

Then! $\frac{\partial}{\partial y}\mu P = \mu'(y)(y^2 - xy) + \mu(y)(2y - x)$

$\frac{\partial}{\partial x}\mu Q = \mu(y)(y)$ 2

$\mu'(y)(y^2 - xy) + \mu(y)(2y - x) = \mu(y)(y)$

4. (25 points) Solve the following differential equation:

$$(2xe^{\frac{y}{x}} - y)dx + xdy = 0$$

Note that both P and Q are homogeneous equations of degree 1!

$P(tx, ty) = 2(tx)e^{\frac{ty}{tx}} - ty = t(2xe^{\frac{y}{x}} - y) = t \cdot P(x,y)$

$Q(tx, ty) = tx = t(x) = t \cdot Q(x,y)$

Let $y = vx$, so that $dy = xdv + vdx$. +5

Then: $(2xe^{\frac{vx}{x}} - vx)dx + x(xdv + vdx) = 0$ +5

$(2xe^{v} - vx)dx + x^2dv + xvdx = 0$

$(2xe^{v})dx + x^2dv = 0$ +5 Multiply by $\mu = \frac{1}{x^2e^v}$

$\frac{2x}{x^2}dx + \frac{1}{e^v}dv = 0$ +5

$\ln(x^2) - e^{-v} = C$

$\boxed{F = \ln(x^2) - e^{-(y/x)} = C}$ ✓ +5

25.