Fall 2017: Math 33B Midte

This is a closed book test. Do all work on the
Scientific calculator is allowed during th

Grade Table (for teacher use only)

Question	Points	Score
1	25	25
2	25	22
3	25	25
4	25	25
Total:	100	97

1. (25 points) The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 600 mg of Iodine 131, how much of the isotope will be left after 30 days?

25

$N(t) = Ae^{-\lambda T}$ $\quad \frac{dN}{dt} = -\lambda N(t)$

$N(t) = Ae^{-\lambda T}$ $\quad A = 600\,mg$

$300\,mg = (600\,mg)e^{-\lambda(8.04)}$

$\frac{1}{2} = e^{-\lambda(8.04)}$

$\ln\frac{1}{2} = -\lambda(8.04)$

$-\frac{\ln\frac{1}{2}}{8.04} = \lambda$

$\lambda = 0.086212$

$N(30) = (600\,mg)e^{-(0.086212)(30)}$

$N(30) \approx 45.175\,mg$

2. (25 points) A 100-gal tank initially contains 40 gal of pure water. Sugar-water solution containing 2 lb of sugar for each gallon of water begins entering the tank at a rate of 4 gal/min. **After 10 minutes**, a drain is opened at the bottom of the tank, allowing the sugar-water solution to leave the tank at a rate of 2 gal/min. What is the sugar content (lb) in the tank at the precise moment that the tank is full of sugar-water solution?

22

$v(t) = 80 + 2t$, call $t=0$ when 10 min and drain turns on.

tank is full @ $t = 10$

$x(t)$ is the amount of sugar in solution @ time t.

rate in $= 4 \cdot 2$

rate in $= 8$

rate out $= 2 \cdot \frac{x(t)}{80+2t}$

rate out $= \frac{x(t)}{40+t}$

$x(0) = 8 \cdot 10 = 80$

$\frac{dx}{dt} = 8 - \frac{x(t)}{40+t}$ ✓

$\frac{dx}{dt} + \frac{x(t)}{40+t} = 8$

$a(t) = \frac{1}{40+t}$ $\quad f(t) = 8$

$u(t) = e^{\int -a(t)\,dt} = e^{\int \frac{1}{40+t}\,dt} = 40+t$

$\int [x(t)(40+t)]' = \int 8(40+t)\,dt$

$x(t)(30+t) = 320t + 4t^2 + C$

$x(t) = \frac{320t + 4t^2 + C}{30+t}$ ✓

$x(0) = 80 = \frac{C}{30}$

$C = 2400$

$x(10) = \frac{320(10) + 4(10)^2 + 2400}{30+10}$

$= \frac{3200 + 400 + 2400}{40}$

$= \frac{6000}{40} =$ 150 lbs of sugar when tank is full

3. (25 points) Solve the following differential equation:

$$(y^2 - xy)dx + (xy - 1)dy = 0$$

$P(x,y) = (y^2 - xy)$ $\quad Q(x,y) = (xy - 1)$

$\frac{\partial P}{\partial y} = 2y - x$ $\quad \frac{\partial Q}{\partial x} = y$

so, not exact, must find integrating factor

$g(y) = \frac{1}{P}(y - x) = \frac{1}{y(y-x)}(y-x) = \frac{1}{y}$, so this is a function of y only

$u(y) = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = y^{-1} = \frac{1}{y}$ ✓

$u(y)P(x,y) = y - x$ $\quad u(y)Q(x,y) = x - \frac{1}{y}$ ✓

Now, this is exact and we can solve

$\int u(y)P(x,y)\,dx = \int y - x\,dx = yx - \frac{1}{2}x^2 + \phi(y)$

$\frac{\partial}{\partial y}(yx - \frac{1}{2}x^2 + \phi(y)) = x + \phi'(y)$ $\therefore \phi'(y) = -\frac{1}{y}$ ✓

$\int \phi'(y)dy = \int -\frac{1}{y} dy$

$\phi(y) = -\ln y$ ✓

$\therefore$ $\boxed{F(x,y) = yx - \frac{1}{2}x^2 - \ln y = C}$ ✓, and we can verify by taking both partial derivatives that this is the correct answer

Great job!

25/25

4. (25 points) Solve the following differential equation:

$$(2xe^{\frac{y}{x}} - y)dx + xdy = 0$$

$P(x,y)$ and $Q(x,y)$ are homogeneous of the same degree so we substitute $y = vx$ and $dy = vdx + xdv$. +5

$F(x, vx) = (2xe^{v} - vx)dx + x(vdx + xdv) = 0$ +5

$2xe^{v}dx - \cancel{vxdx} + \cancel{vxdx} + x^2dv = 0$

$2xe^{v}dx = -x^2dv$ +5

$\int -\frac{2}{x}dx = \int e^{-v}dv$ +5

C sure

$-2\ln x + \cancel{C} = -e^{-v}$ + $v = \frac{y}{x}$

$\boxed{\ln(Cx^{-2}) = -e^{-\frac{y}{x}}}$ 25/25