1. (25 points) The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 600 mg of Iodine 131, how much of the isotope will be left after 30 days?

25

$I(t) = C(1/2)^{Kt}$

$I(0) = 600 = C(1/2)^{0}$

$C = 600$

$I(t) = 600(1/2)^{Kt}$

$\frac{600}{2} = 600(1/2)^{8.04K}$

$\frac{1}{2} = (1/2)^{8.04K}$, $K = \frac{1}{8.04}$

$I(t) = 600(1/2)^{t/8.04}$

$I(30) = 600(1/2)^{30/8.04}$

$I(30) = 45.176$ mg

2. (25 points) A 100-gal tank initially contains 40 gal of pure water. Sugar-water solution containing 2 lb of sugar for each gallon of water begins entering the tank at a rate of 4 gal/min. After 10 minutes, a drain is opened at the bottom of the tank, allowing the sugar-water solution to leave the tank at a rate of 2 gal/min. What is the sugar content (lb) in the tank at the precise moment that the tank is full of sugar-water solution?

First 10 mins) $y' = \text{rate in} - \text{rate out}$ $V(t) = 40 + 4t$

$y' = 2(4) = 8$

$\int dy = \int 8\,dt$

$y(t) = 8t + C$

$y(0) = 0 = 8(0) + C$

$C = 0$

$y(t) = 8t$ ✓

$y(10) = 80$

After 10 mins) $y' = \text{rate in} - \text{rate out}$ $V(t) = 80 + 2t$

$y' = 2(4) - 2\left(\frac{y(t)}{V(t)}\right)$ ✓ $100 = 80 + 2t$

$y' = 8 - \frac{2y}{80+2t}$ $t = 10$ ⇐ tank full

$y' + \frac{y}{40+t} = 8$

$\mu(t) = e^{\int \frac{1}{40+t}dt} = e^{\ln|40+t|} = 40+t$ ✓

$(40+t)\left(y' + \frac{y}{40+t}\right) = 8(40+t)$

$(40+t)y = \int (320+8t)\,dt$

$(40+t)y = 320t + 4t^2 + C$

$y = \frac{4t^2+320t}{40+t} + \frac{C}{40+t}$

$y(10) = \frac{4(100)+320(10)}{40+10}$

$y(10) = 72$ lbs

↑ when tank is full

3. (25 points) Solve the following differential equation:

$$\underbrace{(y^2 - xy)}_{P}dx + \underbrace{(xy - 1)}_{Q}dy = 0$$

$\frac{\partial P}{\partial y} = 2y - x \neq \frac{\partial Q}{\partial x} = y$ Not exact.

$h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xy-1}(2y - x - y)$

$= \frac{y-x}{xy-1}$

$g = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{y^2 - xy}(y - 2y + x)$

$= \frac{-(y-x)}{y(y-x)} = \frac{-1}{y}$

$\mu(y) = e^{-\int \frac{1}{y}dy} = e^{\ln|\frac{1}{y}|} = \frac{1}{y}$

$\frac{(y^2 - xy)}{y}dx + \frac{(xy-1)}{y}dy = \frac{0}{y}$

$\underbrace{(y-x)}_{M}dx + \underbrace{(x - \frac{1}{y})}_{N}dy = 0$

$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x} = 1$ Exact.

$F(x,y) = \int (y - x)dx$

$F(x,y) = yx - \frac{1}{2}x^2 + \phi(y)$

$\frac{\partial F}{\partial y} = x + \phi'(y)$

$= x - \frac{1}{y}$

$\int \phi'(y) = \int -\frac{1}{y}$

$\phi(y) = -\ln|y| + C$

$$\boxed{F(x,y) = yx - \frac{1}{2}x^2 - \ln|y| = C}$$

Great job!

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4. (25 points) Solve the following differential equation:

$$(2xe^{\frac{y}{x}} - y)dx + xdy = 0$$

degree 1 degree 1

Homogenous. ✓

$(2xe^{\frac{y}{x}} - y)dx + xdy = 0$

Let $y = vx$, $dy = vdx + xdv$, $v = \frac{y}{x}$ +5

$(2xe^{v} - vx)dx + x(vdx + xdv) = 0$ +5.

$(2e^{v} - v)dx + (vdx + xdv) = 0$

$(2e^{v} - v + v)dx + xdv = 0$

$\frac{2e^{v}dx}{x2e^{v}} + \frac{xdv}{x2e^{v}}$ +5

$\int\left(\frac{1}{x}dx\right) + \int\left(\frac{1}{2e^{v}}dv\right) = \int 0$ +5.

$$\ln|x| + \frac{1}{2}\int \frac{1}{e^{v}}dv = C$$

$-\frac{1}{2}e^{-v}$

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$u = \frac{1}{e^{v}}$ $v = v$

$du = \frac{-1}{e^{v-1}}$ $dv = 1$

$\frac{1}{2}\int \frac{1}{e^{v}}dv = \frac{1}{2}\left(\left(\frac{1}{e^{v}}\right)v - \int -v\right)$