Fall 2017: Math 33B Midterm - I

This is a closed book test. Do all work on the sheets provided.
Scientific calculator is allowed during the exam.

Grade Table (for teacher use only)

Question	Points	Score
1	25	25
2	25	25
3	25	25
4	25	25
Total:	100	100

1. (25 points) The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 600 mg of Iodine 131, how much of the isotope will be left after 30 days?

25

let $N(t) = Ae^{-\lambda t}$ be the mg of isotope left after t days

$N(0) = A$

$N(t_{1/2}) = \frac{1}{2}A = Ae^{-\lambda t_{1/2}}$

$\frac{1}{2} = e^{-\lambda t_{1/2}}$

$\ln\frac{1}{2} = -\lambda t_{1/2}$

$t_{1/2} = \frac{\ln 1/2}{-\lambda} = \frac{\ln 2}{\lambda}$

$8.04 = \frac{\ln 2}{\lambda}$

$\lambda = \frac{\ln 2}{8.04}$

$N(t) = Ae^{-\lambda t}$

$N(0) = 600 \text{ mg} = Ae^{0}$

$A = 600 \text{ mg}$

$N(t) = 600e^{-\frac{\ln 2}{8.04}t}$

$N(30) = 600e^{-\frac{30}{8.04}\ln 2}$

$= \boxed{600 \cdot 2^{(-30/8.04)} \text{ mg of isotope}}$

2

2. (25 points) A 100-gal tank initially contains 40 gal of pure water. Sugar-water solution containing 2 lb of sugar for each gallon of water begins entering the tank at a rate of 4 gal/min. **After 10 minutes**, a drain is opened at the bottom of the tank, allowing the sugar-water solution to leave the tank at a rate of 2 gal/min. What is the sugar content (lb) in the tank at the precise moment that the tank is full of sugar-water solution?

let $X(t)$ be the lbs of sugar in the tank after $t+10$ minutes
let $V(t)$ be the volume of solution in the tank after $t+10$ minutes
$V(t) \leq 100$

$$X(0) = (2\tfrac{lb}{gal})(4\tfrac{gal}{min})(10\,min) = 80(lb)$$
$$V(0) = 40gal + (4\tfrac{gal}{min})(10\,min) = 80\,gal$$

$$X'(t) = (2\tfrac{lb}{gal})(4\tfrac{gal}{min}) - \frac{X(t)}{V(t)}(2\tfrac{gal}{min})$$
$$V(t) = 80 + 4t - 2t = 80 + 2t$$

$$X' = 8 - \frac{2X}{80+2t} = 8 - \frac{X}{40+t}$$

$$X' + \left(\frac{1}{40+t}\right)X = 8$$

integrating factor: $u = e^{\int \frac{1}{t+40}dt} = e^{\ln(t+40)} = t+40$

$$u\left(X' + \left(\frac{1}{40+t}\right)X\right) = 8(t+40)$$
$$(uX)' = 8(t+40)$$
$$uX = \int 8(t+40)\,dt$$
$$(t+40)X = 4t^2 + 320t + C$$
$$X(t) = \frac{4t^2+320t+C}{t+40}$$

$\Rightarrow$ initial condition

$$X(0) = 80$$
$$80 = \frac{C}{40}$$
$$C = 3200$$

$$X(t) = \frac{4t^2+320t+3200}{t+40}$$

Tank is full when $V(t) = 100$
$$80 + 2t = 100$$
$$t = 10$$

$$X(10) = \frac{4(10)^2 + 320(10) + 3200}{10+40}$$
$$= \frac{400 + 3200 + 3200}{50}$$
$$= \boxed{136\,lb}$$

136
50 | 6800
-50
180
-150
300
-300
0

3. (25 points) Solve the following differential equation:

$$(y^2 - xy)dx + (xy - 1)dy = 0$$

not exact $\quad \frac{\partial}{\partial y}(y^2 - xy) \neq \frac{\partial}{\partial x}(xy-1)$

$$2y - x \neq y$$

~~integrating factor~~ $\mu = \mu(x)$

$\frac{\partial}{\partial y}(\mu(x)(y^2 - xy))$

$\frac{d\mu(x)}{\mu(x)} = \frac{y-x}{xy-1}\, dx$

integrating factor $\mu = \mu(y)$ ✓

$$\mu(y)(2y-x) + \mu'(y)(y^2-xy) = \mu(y)\cdot y$$

$$\mu'(y)(y^2-xy) = \mu(y)(x-y)$$

$$\int \frac{d\mu(y)}{\mu(y)} = \int \frac{x-y}{y^2-xy}\, dy$$

$$\ln \mu = \int \frac{x}{y(y-x)} - \frac{y}{y^2-xy}\, dy$$

$$\ln \mu = \int \left(\frac{1}{y-x} - \frac{1}{y} - \frac{1}{y-x} \right) dy$$

$$\ln \mu = \ln(y-x) - \ln y - \ln|y-x|$$

$$\ln \mu = \ln \frac{y-x}{y(y-x)}$$

$$\mu = \frac{1}{y}$$ ✓

$\int \mu(y^2-xy)\,dx + \mu(xy-1)\,dy = 0$

Great job!

25/25

$$(y-x)dx + \left(x - \frac{1}{y}\right)dy = 0$$

$$F(x,y) = \int y - x \, dx + \phi(y)$$

$$= yx - \frac{1}{2}x^2 + \phi(y)$$

$$\frac{\partial F}{\partial y} = x - \frac{1}{y} = x + \phi'(y)$$

$$\phi'(y) = -\frac{1}{y} \quad \rightarrow \quad \phi(y) = -\ln y + C$$

$$\boxed{F(x,y) = yx - \tfrac{1}{2}x^2 - \ln y = C}$$

4. **(25 points)** Solve the following differential equation:

$$(2xe^{\frac{y}{x}} - y)dx + xdy = 0$$

$$2(tx)e^{\frac{ty}{tx}} - ty = t(2xe^{\frac{y}{x}} - y)$$

$$tx = tx$$

This equation is homogeneous (both of degree 1)

$$y = vx$$

$$(2xe^{\frac{vx}{x}} - vx)dx + x(vdx + xdv) = 0$$

$$\frac{x(2e^{v} - v)dx + \cancel{x}(vdx + xdv)}{x} = \frac{0}{x}$$

$$(2e^{v})dx + xdv = 0$$

Integrating factor: $u = \frac{1}{x(2e^{v})}$

$$\frac{(2e^{v})dx + xdv}{u} = \frac{0}{u}$$

$$\frac{dx}{x} + \frac{1}{2e^{v}}dv = 0$$

$$\int \frac{dx}{x} = \int -\frac{1}{2}e^{-v}\,dv$$

$$\ln x = \frac{1}{2}e^{-v} + c$$

$$v = \frac{y}{x}$$

$$\ln x = \frac{1}{2}e^{-\frac{y}{x}} + C_1$$

$$2(\ln x - C_1) = e^{-y/x} \qquad C = -2C_1$$

$$\ln(2\ln x + C) = \ln e^{-y/x}$$

$$-\frac{y}{x} = \ln(2\ln x + C)$$

$$\boxed{y = -x\ln(2\ln x + C)}$$