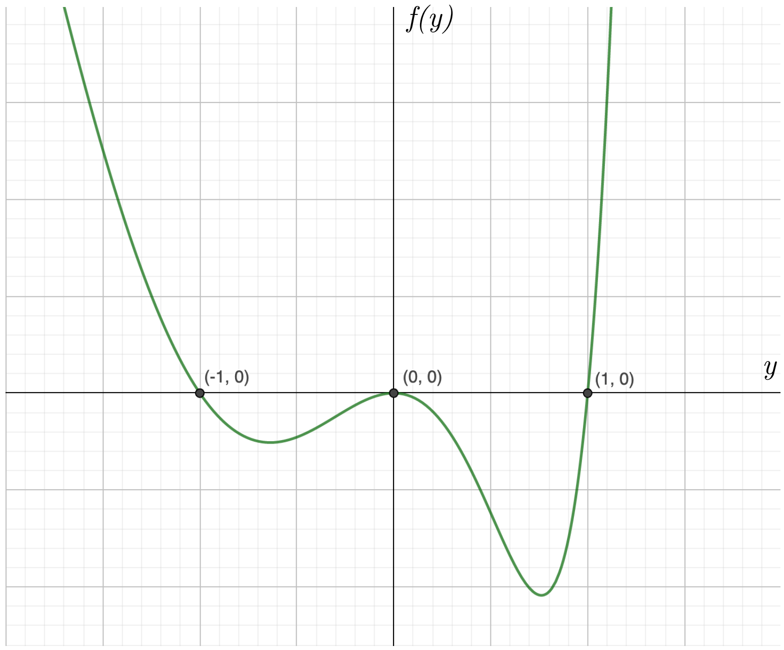
MATH 33B: MIDTERM 2

Question 1.
Consider the autonomous differential equations $y' = y(y-1)f(y)$ with the function $f(y)$ having the graph below.

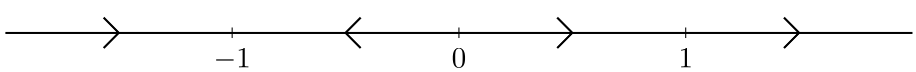


(1) *Construct the phase line and classify each equilibrium point as either unstable or asymptotically stable.*
(2) *Sketch the equilibrium solutions in the ty-plane. These equilibrium solutions divide the ty-plane into regions. Sketch at least one solution curve in each of these regions.*
(3) *Predict the limiting behavior of the solution to the initial value problem*

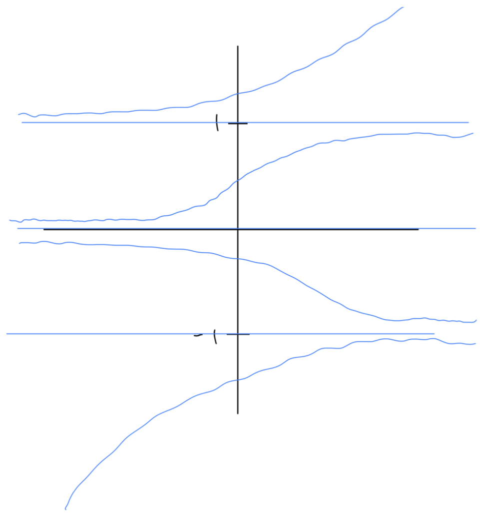
$$y' = y(y-1)f(y), \quad y(0) = y_0$$

depending on the initial value y_0. Make sure to consider all possible cases!

Solution:
(1) The equilibrium points occur at $y = -1, 0, 1$. The phase line is:



The equilibrium point $y = -1$ is stable; the other two are not.
(2) From the phase line, we obtain the following possible graphs of $y(t)$:



(3) The initial condition is $y(0) = y_0$. By inspecting either the graph or the phase line, we see that

$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & y_0 > 1 \\ 1 & 0 < y_0 \le 1 \\ 0 & y_0 = 0 \\ -1 & y_0 < 0 \end{cases}$$

Question 2.

(1) Find the general solution to the following homogeneous second-order linear differential equation:

$$y'' + 4y' + 4y \;=\; 0.$$

(2) Use variation of parameters to find the solution to the following initial value problem (on the interval $(0, +\infty)$):

(i) $y'' + 4y' + 4y = t^{-2}e^{-2t}$

(ii) $y(1) = 0,\ y'(1) = 0.$

Solution:

(1) The characteristic polynomial associated to this linear differential equation is

$$X^2 + 4X + 4 = (X + 2)^2$$

So we have a repeated root of $X = -2$. This means that a fundamental set of solutions is provided by $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$. The general solution is any linear combination of these two:

$$y = C_1e^{-2t} + C_2te^{-2t}$$

(2) The method of variation of parameters tells us we can find a particular solution $y_P = u_1y_1 + u_2y_2$ where u_1, u_2 are given by the formulas

$$u_1 = -\int \frac{y_2t^{-2}e^{-2t}}{W}$$

and

$$u_2 = \int \frac{y_1t^{-2}e^{-2t}}{W}$$

where W is the Wronskian of y_1, y_2. Computing the Wronskian gives

$$W = y_1 y_2' - y_1' y_2 = e^{-2t}(te^{-2t})' - (e^{-2t})' te^{-2t} = e^{-2t}(e^{-2t} - 2te^{-2t}) + 2e^{-2t} te^{-2t} = e^{-4t}$$

Now we can compute u_1 and u_2:

$$u_1 = -\int \frac{te^{-2t} t^{-2} e^{-2t}}{e^{-4t}} = -\int t^{-1} = -\ln|t|$$

and

$$u_2 = \int \frac{e^{-2t} t^{-2} e^{-2t}}{e^{-4t}} = \int t^{-2} = -\frac{1}{t}$$

(Now we are only interested in the interval $t > 0$ in this problem, so from here on we will omit the absolute value).

Putting these together gives us a particular solution of

$$y_P = -(\ln t)e^{-2t} - e^{-2t}$$

The solution to the initial value problem will be this particular solution modified by a solution to the homogeneous equation, like

$$y = y_P + C_1 e^{-2t} + C_2 te^{-2t}$$

We just need to determine the coefficients C_1, C_2. We have

$$y = -(\ln t)e^{-2t} - e^{-2t} + C_1 e^{-2t} + C_2 te^{-2t}$$

and

$$y' = 2(\ln t)e^{-2t} - \frac{1}{t}e^{-2t} + 2e^{-2t} - 2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 te^{-2t}$$

So from $y(1) = 0$ we get

$$0 = e^{-2}(-1 + C_1 + C_2)$$

and from $y'(1) = 0$ we get

$$0 = e^{-2}(-1 + 2 - 2C_1 + C_2 - 2C_2) = e^{-2}(1 - 2C_1 - C_2)$$

This gives a system of equation

$$C_1 + C_2 = 1$$
$$2C_1 + C_2 = 1$$

Solving however one likes gives $C_1 = 0, C_2 = 1$. This gives us the solution of iniital value problem:

$$y = -(\ln t)e^{-2t} - e^{-2t} + te^{-2t} = (t - \ln t - 1)e^{-2t}$$

Question 3.

(1) *Use the method of undetermined coefficients to find particular solutions to each of the following:*

(a) $y'' - 2y' - 3y = e^{-t}$

(b) $y'' - 2y' - 3y = t$

(2) *Find the general solution to the following:*

$$y'' - 2y' - 3y = \ 2e^{-t} - 3t$$

(Hint: use (1)).

Solution: Note that the characteristic polynomial for $y'' - 2y' - 3y$ is $(x+1)(x-3)$, so the general solution to the homogeneous equation is $Ae^{-t} + Be^{3t}$.

(1) For equation (a), our target function is an exponential and a solution to the homogeneous equation, so our guess should be

$$y(t) = Ate^{-t}.$$

Plugging into equation (a) we get

$$e^{-t} = y'' - 2y' - 3y = A(-2e^{-t} + te^{-t}) - 2A(e^{-t} - te^{-t}) - 3Ate^{-t} = A(-4)e^{-t},$$

so $A = \frac{-1}{4}$.

For equation (b), our target is a polynomial of degree 1, so our guess should be

$$y(t) = At + B.$$

Plugging in, we get

$$t = y'' - 2y' - 3y = -2A - 3(At + B) = -3At - (3B + 2A)$$

So, $A = \frac{-1}{3}$, $B = \frac{2}{9}$.

(2) From (1) and the superposition rule, a particular solution to our equation is

$$\frac{-1}{2}te^{-t} + t - \frac{2}{3}.$$

Combining this with the solution to the homogenous equation gives the general solution

$$y(t) = Ae^{-t} + Be^{3t} + \frac{-1}{2}te^{-t} + t - \frac{2}{3}.$$

Question 4.

(1) *Consider the homogeneous equation*

$$t^2y'' - 5ty' + ky = 0, \qquad \textit{for } t > 0.$$

Determine a real number k such that $y_1(t) = t^3$ is a solution.

(2) *Consider the equation*

$$(*) \qquad t^2y'' - 5ty' + ky = t^3, \qquad \textit{for } t > 0.$$

with the fixed k that you found in part (1).

Look for a solution to $(*)$ of the form $y(t) = v(t)y_1(t)$, where $v(t)$ is an unknown function. That is, plug this substitution into the equation, simplify, and derive a (first-order linear) differential equation for $w(t) = v'(t)$.

(3) *Use part (2) to find the general solution to $(*)$.*

(4) *Find* $y_2(t)$*, which, together with* $y_1(t) = t^3$*, would form a fundamental set of solutions to the homogeneous equation associated with* $(*)$*. Explain!*

Show your work!

Solution. (1) Let $y(t) := t^3$, note that $y'(t) = 3t^2$ and $y''(t) = 6t$. Plugging $y(t)$ into the lefthand side of the equation we see that

$$\begin{aligned} t^2y''(t) - 5ty'(t) + ky(t) &= t^2 \cdot 6t - 5t \cdot 3t^2 + kt^3 \\ &= (6 - 15 + k)t^3 \\ &= (-9 + k)t^3 \end{aligned}$$

Thus, in order to make the LHS $= 0$ when we plug in $y(t) = t^3$, we require $k = 9$.

(2) We will look for a solution of the form $y(t) = v(t)y_1(t) = v(t)t^3$. First we compute the first and second derivatives of $y(t)$:

$$\begin{aligned} y'(t) &= 3vt^2 + v't^3 \\ y''(t) &= 6vt + 3v't^2 + 3v't^2 + v''t^3 \end{aligned}$$

Next we substitute $y(t)$ into the lefthand side of the equation (with $k = 9$):

$$\begin{aligned} t^2y''(t) - 5ty'(t) + 9y(t) &= t^2(6vt + 3v't^2 + 3v't^2 + v''t^3) - 5t(3vt^2 + v't^3) + 9vt^3 \\ &= t^3(6v + 3tv' + 3tv' + t^2v'' - 15v - 5tv' + 9v) \\ &= t^3(t^2v'' + tv') \end{aligned}$$

Since the lefthand side must equal the righthand side, we get the equation:

$$t^3(t^2v'' + tv') = t^3$$

Dividing by t^3 (which is allowed, since we are only considering values $t > 0$), we get the equation:

$$t^2v'' + tv' = 1.$$

Next we divide by t^2:

$$v'' + \frac{1}{t}v' = \frac{1}{t^2}$$

Making the substitution $w(t) = v'(t)$, we get the first-order linear equation:

$$w' + \frac{1}{t}w = \frac{1}{t^2}.$$

(3) We will now solve for $(*)$. First we solve for $w(t)$: compute the integrating factor:

$$\mu(t) = \exp\left(\int \frac{dt}{t}\right) = \exp(\ln|t|) = |t| = t \quad (\text{since } t > 0)$$

and then simplify the first-order equation to:

$$(tw)' = \frac{1}{t}$$

Integrating both sides yields:

$$tw(t) = \ln t + C_1$$

and so

$$w(t) = \frac{\ln t}{t} + C_1\frac{1}{t}$$

Since $v'(t) = w(t)$, we now get the differential equation

$$v'(t) = \frac{\ln t}{t} + C_1\frac{1}{t}.$$

Integrating both sides yields:

$$v(t) = \frac{(\ln t)^2}{2} + C_1 \ln t + C_2.$$

This is the general solution for the second-order equation which $v(t)$ satisfies. To get the general solution to $(*)$ we need to multiply by y_1:

$$y(t) = \frac{t^3(\ln t)^2}{2} + C_1 t^3 \ln t + C_2 t^3.$$

(4) Consider the following two solutions to the inhomogeneous equation:

$$\begin{aligned} z_1(t) &:= \frac{t^3(\ln t)^2}{2} + t^3 \ln t + t^3 \\ z_2(t) &:= \frac{t^3(\ln t)^2}{2} + t^3 \end{aligned}$$

(corresponding to $(C_1, C_2) = (1,1)$ and $(0,1)$, respectively). Then $y_2(t) := z_1(t) - z_2(t) = t^3 \ln t$ is a solution to the homogeneous equation. Since $y_2(t)$ is linearly independent from $y_1(t)$, we conclude that y_1, y_2 is a fundamental set of solutions to the homogeneous equation. $\square$