University of California, Los Angeles
Spring 2020

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MATH 33B: Differential Equations
Final Exam

This exam contains 4 pages (including this cover page) and 8 problems. The following rules apply:

- **You have 24 hours to submit this take home exam**. **The deadline for submission is Saturday, June 13, 8:00am (PDT)**. Please, note that this deadline applies to all students, including those registered with CAE.
- **You should submit your solutions to Gradescope by the deadline.** The first page of your solution should include your name (printed) followed by your UCLA ID (also printed) on the top of the page. Below, please include the following statement, followed by your signature and date (the first page should not contain anything else, i.e., no solutions on the first page):
  "I assert, on my honor, that I have not received assistance of any kind from any other person, and have not used any non-permitted materials or technologies while working on the final. I agree with the rules summarized on the exam assignment cover page"

  It is your responsibility to make sure that the files are uploaded correctly. In case of any technical difficulties you should notify your instructor immediately. In particular, if you have issues with uploading your solutions to Gradescope, you may send your solutions to the instructor to upload (before the deadline). Please, use the following email address: allen@math.ucla.edu
- **Collaborations on the final are not allowed.** You are under strict instructions not to discuss the exam or questions related to the exam with anybody. Please, be reminded of the Student Conduct Code (it can be found at www.deanofstudents.ucla.edu; see, in particular, Section 102.01 on academic dishonesty).
- **You are allowed to use the lecture notes posted online and the textbook**, as well as any other resources that were posted on the course webpages, while working on the exam. **You should not use any other resources, including online ones.**
- **If you use a result from class, discussion session, the textbook, lecture notes, or a homework/midterm, you must indicate this**, reference the source, and explain why the result may be applied.
- **You may use a calculator/computer for arithmetic only**, i.e., simplifying expression which involve $+$, $\times$, $-$, $/$. **You should not use computing systems (e.g., Mathematica or Matlab) while working on the problems.**
- Show your work on each problem. **Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive very little credit.

GOOD LUCK!

1. Suppose $a, b, c \in \mathbb{R}$ are fixed constants such that $a < b < c$. Consider the following initial value problem:

$$y' + \frac{y}{t-c} = \frac{b-c}{(t-c)(t-a)}, \qquad y(b) = 1$$

   (a) (2 points) Determine the interval of existence of the unique solution $y(t)$ (Note: this does not require any computation but may depend on the values of a, b, c).

   (b) (4 points) Compute a valid integrating factor $\mu(t)$.

   (c) (4 points) Compute the unique solution to the initial value problem.

2. Consider the following differential equation:

$$\frac{dy}{dt} = \frac{-2t\cos y - 3t^2 y}{t^3 - t^2 \sin y - y} \qquad (\dagger)$$

   (a) (2 points) Convert (†) into a differential form equation. Determine, with justification, whether your differential form equation is exact.

   (b) (4 points) Determine the general solution of (†). You may leave your answer in implicit form.

   (c) (2 points) Solve the following initial value problem:

$$\frac{dy}{dt} = \frac{-2t\cos y - 3t^2 y}{t^3 - t^2 \sin y - y}, \qquad y(4) = 0.$$

   You may leave your answer in implicit form and you do not need to specify the interval of existence.

3. Consider the following matrix:

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$$

   (a) (3 points) Find all the eigenvalues of the matrix A and for each eigenvalue determine a basis of the associated eigenspace.

   (b) (1 point) Does A have an eigenbasis? Yes or no (no justification needed).

   (c) (3 points) Determine the general solution to following linear system:

$$\mathbf{x}' = A\mathbf{x}$$

   (d) (3 points) Determine the specific solution to the following initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

4. Consider the following initial value problem:

$$y' = \sin(y+t) + e^t - \sin(e^t + t), \qquad y(0) = 2$$

(a) (2 points) Show that there exists an interval $I \subseteq \mathbb{R}$ such that $0 \in I$, and a differentiable function $y : I \to \mathbb{R}$ which is a solution to this initial value problem.

(b) (3 points) Is the solution in (a) unique? Explain why or why not.

(c) (3 points) One of the following three things is true:

1. $y(t) > e^t$ for all $t \in I$
2. $y(t) < e^t$ for all $t \in I$
3. Neither 1. nor 2.

State which of the above is true and justify your answer.

5. This is a mixing problem with three tanks, and five pipes. You will not be required to solve the mixing problem, just to set it up correctly. The initial conditions of the tanks are:

- Tank 1 begins with 80 gal of solution and 10 lbs of salt.
- Tank 2 begins with 70 gal of solution and 8 lbs of salt.
- Tank 3 begins with 60 gal of solution and 6 lbs of salt.

The pipes are configured as follows:

- There is a pipe pumping in 10 gal/min of pure water into Tank 1.
- There is a pipe which connects Tank 1 to Tank 2 and transfers solution at a rate of 10 gal/min.
- There is a pipe which connects Tank 2 to Tank 3 and transfers solution at a rate of 30 gal/min.
- There is a second pipe which connects Tank 3 back to Tank 2 and transfers solution at a rate of 20 gal/min.
- There is a pipe which drains solution out of Tank 3 at a rate of 10 gal/min.

(a) (2 points) Draw and label a diagram illustrating the configuration of this mixing problem. The diagram should include all pipes, all flow rates and flow directions, and the inital conditions of the three tanks.

(b) (4 points) Let $y_1(t), y_2(t), y_3(t)$ be the amount of salt (in lbs, not lbs/gal) in each of tanks 1, 2, and 3 at time t. Set up but do not solve a linear system initial value problem which has the triple of functions $y_1(t), y_2(t), y_3(t)$ as its unique solution.

6. Consider the inhomogeneous second-order linear differential equation:

$$y'' + 4y' + 4y \; = \; t^{-2}e^{-2t}, \quad t > 0$$

   (a) (3 points) Find a fundamental set of solutions to the associated *homogeneous* second-order linear differential equation.

   (b) (4 points) Find a particular solution to the *inhomogeneous* equation.

   (c) (1 point) Write down the general solution to the *inhomogeneous* equation.

7. Suppose $a \in \mathbb{R}$ is a fixed constant such that $a > 1$. Consider the following autonomous differential equation:

$$y' \; = \; (y+1)(y^2 - a)$$

   (a) (3 points) Draw the corresponding phase line. Be sure to fully label the diagram which includes indicating the equilibrium points and whether they are asymptotically stable or unstable.

   (b) (3 points) Sketch the corresponding direction field. Include solution curves for all constant solutions and in each region between constant solutions (and above the largest constant solution and below the lowest constant solution) include a solution curve.

   (c) (3 points) Consider the initial value problem

$$y' \; = \; (y+1)(y^2 - a), \qquad y(0) \; = \; y_0$$

   where $y_0 \in \mathbb{R}$ is an arbitrary but fixed constant. Suppose $y(t)$ is the unique solution. What is $\lim_{t\to+\infty} y(t)$? Your answer should include *all* possible cases depending on the particular value of y_0.

8. Consider the following inhomogeneous second-order linear differential equation:

$$y'' - 2y' + 5y \; = \; t^2$$

   (a) (3 points) Find the general solution to the associated *homogeneous* second-order linear differential equation.

   (b) (4 points) Find a particular solution to the *inhomogeneous* equation.

   (c) (1 point) Write down the general solution to the *inhomogeneous* equation.