Fall 2017: Math 33B Midterm - II

This is a closed book test. Do all work on the sheets provided.

Grade Table (for teacher use only)

Question	Points	Score
1	25	25
2	25	25
3	25	19
4	25	21
Total:	100	90

1. (25 points) Solve the equation system below with initial conditions by steps (a) - (c).

25

$$\begin{cases} x'(t) = 2x(t) - y(t) \\ y'(t) = x(t) \end{cases} \quad \text{with} \quad \begin{cases} x(0) = 1 \\ y(0) = 2 \end{cases}$$

(a) (5 points) Show the second order equation that is satisfied by $y(t)$;

(b) (5 points) Show the corresponding initial conditions of this equation;

(c) (10 points) Solve the function $y(t)$ from the previous step;

(d) (5 points) Solve the function $x(t)$.

(a)

$x(t) = y'(t)$

$x'(t) = y''(t) = 2x(t) - y(t)$

$y''(t) = 2y'(t) - y(t)$

$y''(t) - 2y'(t) + y(t) = 0$

(b)

$x(0) = y'(0) = 1$

$y(0) = 2$

(c)

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

$y(t) = C_1 e^t + C_2 t e^t$ $\quad y'(t) = C_1 e^t + C_2 e^t + C_2 t e^t$

$y(0) = C_1 = 2$ $\quad y'(0) = C_1 + C_2 = 1$

$C_1 = 2, \; C_2 = -1$

$y(t) = 2e^t - te^t$

(d)

$x(t) = y'(t)$

$x(t) = 2e^t - e^t - te^t$

$= e^t - te^t$

2. (25 points) A 0.1-kg mass is attached to a spring having a spring constant 3.6 kg/s². The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present. Let $x(t)$ be the displacement of the mass at t.

(a) (5 points) Show the differential equation satisfied by $x(t)$ and its initial conditions.

(b) (5 points) Solve the function $x(t)$.

(c) (5 points) What is the **amplitude** of the motion?

(d) (5 points) What is the **frequency** of the motion?

(e) (5 points) What is the **phase** of the motion?

(a) +5

$m = 0.1$ kg

$k = 3.6$ kg/s²

$v(t) = x'(t)$ $\quad v(0) = 0.4$

$x = 0$

$mx''(t) + kx(t) = 0$

$0.1x''(t) + 3.6x(t) = 0 \qquad x(0) = 0 \qquad x'(0) = 0.4$

(b) +5

$x''(t) + 36x(t) = 0$

$\lambda^2 + 36 = 0 \quad \lambda_1 = 6i, \lambda_2 = -6i$

$x(t) = C_1\cos 6t + C_2\sin 6t$

$x(0) = C_1 = 0$

$x'(t) = -6C_1\sin 6t + 6C_2\cos 6t$

$x'(0) = 6C_2 = 0.4$

$C_2 = \frac{1}{15}$

$x(t) = \frac{1}{15}\sin 6t$

(c) +5 amplitude: $A = \sqrt{a^2+b^2}$

$A = \frac{1}{15}$

(d) +5 frequency $\omega = 6$ rad/s

25/25

5 (e) $b = A\sin\phi$

$\frac{b}{A} = \sin\phi$

$\sin\phi = 1 \qquad \phi = \frac{\pi}{2}$

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3\. (25 points) Solve the following differential equation by steps.

$$y'' + 4y' + 4y = 5e^{-2t} + 2\sin(2t) + 3t + 4 \qquad (1)$$

(a) (5 points) Find the general solution to the associated homogeneous equation.

(b) (5 points) Find a particular solution $y_{p_1}(t)$ to the equation: $y'' + 4y' + 4y = e^{-2t}$.

(c) (5 points) Find a particular solution y_{p_2} to the equation: $y'' + 4y' + 4y = \sin(2t)$.

(d) (5 points) Find a particular solution y_{p_3} to the equation: $y'' + 4y' + 4y = 3t + 4$.

(e) (5 points) Find the expression of the general solution $y(t)$ to the equation (1).

(a) 5

$y_h'' + 4y_h' + 4y_h = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$

$\lambda = -2$

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$y_h(t) = C_1 e^{-2t} + C_2 t e^{-2t}$ ✓

(b) $y'' + 4y' + 4y = e^{-2t}$

$y_{p_1} = ae^{-2t}$

$y_{p_1}' = -2ae^{-2t}$

$y_{p_1}'' = 4ae^{-2t}$

$y_{p_1}'' + 4y_{p_1}' + 4y_{p_1} = 4ae^{-2t} + 4(-2ae^{-2t}) + 4(ae^{-2t}) = 0$

$\lambda^2 + 4\lambda + 4 = 0$ $\lambda = -2$

-2 is repeated root of the homogeneous eq.

$y_{p_1} = t^2 ae^{-2t}$ $\quad y_{p_1}' = 2tae^{-2t} - 2t^2ae^{-2t}$ $\quad y_{p_1}'' = 2ae^{-2t} - 4tae^{-2t} - 4tae^{-2t} + 4t^2ae^{-2t}$

$= 2ae^{-2t} - 8tae^{-2t} + 4t^2ae^{-2t}$

$y_{p_1}'' + 4y_{p_1}' + 4y_{p_1} = (2ae^{-2t} - 8tae^{-2t} + 4t^2ae^{-2t}) + 4(2tae^{-2t} - 2t^2ae^{-2t}) + 4t^2ae^{-2t}$

$= ae^{-2t}(2 + t(-8+8) + t^2(4-8+4))$

$= 2ae^{-2t} = e^{-2t}$ $\quad a = \frac{1}{2}$

$\boxed{y_{p_1} = \frac{1}{2}t^2e^{-2t}}$ ✓

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(c) $y'' + 4y' + 4y = \sin 2t$ $\quad y_{p_2} = a\cos 2t + b\sin 2t$ $\quad y_{p_2}' = -2a\sin 2t + 2b\cos 2t$ $\quad y_{p_2}'' = -4a\cos 2t - 4b\sin 2t$

$y_{p_2}'' + 4y_{p_2}' + 4y_{p_2} = (-4a\cos 2t - 4b\sin 2t) + 4(-2a\sin 2t + 2b\cos 2t) + 4(a\cos 2t + b\sin 2t)$

$= \cos 2t(-4a + 8b + 4a) + \sin 2t(-4b + 8a + 4b)$

$= 8b\cos 2t + 8a\sin 2t = \sin 2t$ ✓

$a = 1/8 \quad b = 0$

$\boxed{y_{p_2} = -\frac{1}{8}\cos 2t}$

4. (25 points) Solve the general solution of the following equation by steps.

$$y'' + y' + y = 2t\sin(t) \tag{2}$$

(a) (10 points) Find constants A and B, such that the function $z(t) = (At+B)e^{it}$ solves the equation: $z'' + z' + z = te^{it}$.

(b) (10 points) Find a particular solution $y_p(t)$ to the equation (2).

(c) (5 points) Find the general solution to the equation (2).

(a) $z(t) = (At+B)e^{it}$ $z'(t) = i(At+B)e^{it} + Ae^{it} = e^{it}(A + Ait + Bi)$

$z''(t) = ie^{it}(A + Ait + Bi) + Aie^{it}$

$= ie^{it}(2A + Ait + Bi)$ ✓

$z'' + z' + z = ie^{it}(2A + Bi + Ait) + e^{it}(A + Ait + Bi) + (At+B)e^{it}$

$= e^{it}(-B - At + 2Ai + Ait + A + Bi + At + B)$

$= e^{it}(i(2A + At + B) + A)$

$= e^{it}((Ai)t + (2Ai + Bi + A)) = te^{it}$

$Ai = 1$ $2Ai + Bi + A = 0$

$A = \frac{1}{i}$ ✓ $2 + Bi + \frac{1}{i} = 0$

$B = (-\frac{1}{i} - 2)\frac{1}{i} = 1 - \frac{2}{i}$

$z(t) = (\frac{1}{i}t + 1 - \frac{2}{i})e^{it}$ $B = 1 - \frac{2}{i}$ ✓

10/10

(b) $2t\sin t = 2t\,\mathrm{Im}(\cos t + i\sin t) = 2t\,\mathrm{Im}(e^{it})$

$y_p = \mathrm{Im}(z(t))$ $z(t) = (i^{-1}(t-2)+1)(\cos t + i\sin t)$

$= (t-2)\sin t$ $= i^{-1}(t-2)\cos t + (t-2)\sin t + \cos t + i\sin t$

$y_p = 2\mathrm{Im}(z(t))$ $= i(\sin t - (t-2)\cos t) + (t-2)\sin t + \cos t$

$y_p = -((t-2)\cos t + \sin t)2$ Almost. 8/10

(c) $y'' + y' + y = 0$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{3}i}{2}$ $y_1 = e^{-t}\cos\sqrt{3}t$ $y_2 = e^{-t}\sin\sqrt{3}t$

3/5. 21/25

$y(t) = C_1 e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + C_2 e^{-t}\sin\sqrt{3}t - (t-2)\cos t + \sin t$