

Math 33B
Winter 2016
Midterm
02/08/16
Time Limit: 50 Minutes

Name (Print): ~~XXXXXXXXXX~~

TA name Alex section number 1A

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem. **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit. Formula that you might or might not need:

- The differential form $Pdx + Qdy$ has an integrating factor depending on one of the variables under the following conditions.
If $h = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of x only, then $\mu = e^{\int h(x)dx}$ is an integrating factor.
If $g = \frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of y only, then $\mu = e^{-\int g(y)dy}$ is an integrating factor.

- $e^{i\theta} = \cos \theta + i \sin \theta$
- $\cos(2\theta) = 2 \cos^2 \theta - 1$

Do not write in the table to the right.

Problem	Points	Score
1	20	17
2	20	2
3	20	15
4	20	13
5	20	1
Total:	100	48

- 17 1. (20 points) Solve the equation $\frac{dy}{dx} = \frac{e^{-y} \cos^2 x}{1+y}$ with $y(0) = 1$. Justify your answer. $x=0$

$$(1+y)e^y dy = \cos^2 x dx$$

$$u=1+y \quad dv=e^y$$

$$du=1 \quad v=e^y$$

$$= \int \left(\frac{\cos 2x}{2} + \frac{1}{2} \right) dx$$

$$(1+y)e^y - \int e^y$$

$$= \frac{1}{4} \sin 2x + \frac{x}{2}$$

$$(1+y)e^y - e^y =$$

$$\boxed{(1+y)e^y - e^y = \frac{1}{4} \sin 2x + \frac{x}{2} + C}$$

$y e^y$

$$C = ? - 3$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2x) = 2\cos^2x - 1$$

$$\frac{\cos 2x + 1}{2} = \cos^2 x$$

2. (20 points) Solve the equation $\frac{dy}{dx} = \frac{y + xe^{-\frac{y}{x}}}{x}$. Justify your answer.

$$x dy = (y + xe^{-\frac{y}{x}}) dx$$

$$0 = \underbrace{(y + xe^{-\frac{y}{x}})}_p dx + \underbrace{(-x)}_Q dy$$

$$\frac{dp}{dy} = (1 + x(-\frac{1}{x}))e^{-\frac{y}{x}} = 1 - e^{-\frac{y}{x}}$$

$$\frac{dQ}{dx} = -1 \quad \frac{dp}{dy} - \frac{dQ}{dx} = 1 - e^{-\frac{y}{x}} - (-1) = 2 - e^{-\frac{y}{x}}$$

$$h = \frac{2 - e^{-\frac{y}{x}}}{-x}$$

$$g = \frac{2 - e^{-\frac{y}{x}}}{y + xe^{-\frac{y}{x}}}$$

$$= -\frac{2}{x} + \frac{1}{xe^{\frac{y}{x}}}$$

$$= \frac{2 - e^{\frac{y}{x}}}{y + \frac{x}{e^{\frac{y}{x}}}}$$

$$= \frac{2e^{\frac{y}{x}} - 1}{ye^{\frac{y}{x}} + x}$$

$$h = \frac{1}{2} \left(\frac{dp}{dy} - \frac{dQ}{dx} \right)$$

$$g = \frac{1}{p} \left(\frac{dp}{dy} - \frac{dQ}{dx} \right)$$

3. (a) (10 points) Solve the equation $y'' + 6y' + 9y = 0$. Here $y = y(t)$ is a function of t . Justify your answer.

10

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

$$\lambda = -3$$

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

5

- (b) (10 points) Solve the equation $y'' + 6y' + 9y = e^{-3t} + t^2$. Justify your answer.

$$y_p = a e^{-3t} + b t^2 + c t + d$$

$$6 y' = (-3a e^{-3t} + 2b t + c)$$

$$9 y'' = (9a e^{-3t} + 2b)$$

$$a e^{-3t} + b t^2 - 24a e^{-3t} + 12b t + 81a e^{-3t} + 18b = e^{-3t} + t^2$$

$$58a e^{-3t} + b t^2 + 12b t + 18b = e^{-3t} + t^2$$

$$58a e^{-3t} + b t^2 + c t + 2b t + 6c + 2b = t^2$$

$$a = \frac{1}{58}$$

$$b t^2 + c t + 2b t + 6c + 2b = t^2$$

$$t^2 + c t + 2t + 6c + 2 = t^2$$

$$c t + 2t + 6c + 2 = 0$$

$$c = -2 \quad -2t + 2t + 6(-2) + 2 = 0$$

$$b = 1$$

$$c = -2$$

$$d = -10$$

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t} + \frac{1}{58} a e^{-3t} + t^2 - 2t - 10$$

$$f'(t) = e^{2t} + 2te^{2t}$$

4. Let $f(t) = te^{2t}$ and $g(t) = t^2$ be two functions of t .

(a) (4 points) Compute the Wronskian of f and g .

$$\begin{aligned} w(t) &= \det \begin{vmatrix} te^{2t} & t^2 \\ e^{2t} + 2te^{2t} & 2t \end{vmatrix} = (te^{2t})(2t) - (e^{2t} + 2te^{2t})t^2 \\ &= 2t^2e^{2t} - t^2e^{2t} - 2t^3e^{2t} \\ &= te^{2t} - 2t^3e^{2t} \\ &= e^{2t}(t - 2t^3) = e^{2t}(t)(t - 2t^2) \end{aligned}$$

(b) (8 points) Show that f and g are linearly independent functions of t .

$$\begin{aligned} te^{2t} &= c t^2 && \text{let } c \in \mathbb{R} \\ \frac{e^{2t}}{t} &= c \end{aligned}$$

c is not a constant multiple, so

$f \neq cg$ and therefore not linearly independent

(c) (8 points) Show that there do not exist differentiable functions $p(t)$ and $q(t)$ such that $y(t) = c_1f(t) + c_2g(t)$ is the general solution to $y'' + p(t)y' + q(t)y = 0$, where c_1 and c_2 are arbitrary constants.

$$\begin{aligned} y(t) &= c_1te^{2t} + c_2t^2 \\ y'(t) &= c_1(e^{2t} + 2te^{2t}) + 2c_2t \\ y''(t) &= c_1(2e^{2t} + 4te^{2t} + 2e^{2t}) + 2c_2 \\ &= c_1(4e^{2t} + 4te^{2t}) + 2c_2 \\ c_1(4e^{2t})(1+t) + 2c_2 + p(t)[c_1(e^{2t} + 2te^{2t}) + 2c_2t] + q(t)[c_1te^{2t} + c_2t^2] &= 0 \end{aligned}$$

5. $\frac{dy}{dt} = 3y^{\frac{2}{3}}(y-1)(y^2-9)\cos y$

(a) (10 points) Find all the constant solutions that are (asymptotically) stable. Justify your answer.



$3y^{\frac{2}{3}}(y-1)(y^2-9)\cos y$

$y < -3 \quad - \quad +$

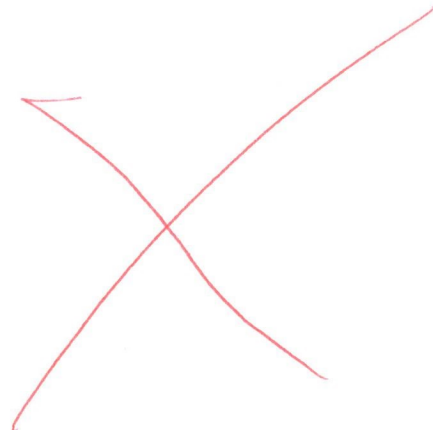
$-3 < y < -\frac{3}{2} \quad - \quad -$



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(b) (10 points) Are the solutions to the above equation together with the initial condition $y(0) = 1$ unique? Justify your answer.



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